

A generalization of simple Harnack curves

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Abstract

Simple Harnack curves, introduced in [Mik00], are smooth real algebraic curves in maximal position in toric surfaces. In the present paper, we suggest a natural generalization of simple Harnack curves by relaxing the smoothness assumption. After mentioning some of their properties, we address the question of their construction. We define tropical Harnack curves and show that their approximation using Mikhalkin's machinery produces many new example of simple Harnack curves. We determine the topological classification of simple Harnack curves with a hyperbolic node in the fashion of [Mik00], and show that the space of such curves can be locally parametrized by the space of their tropical avatars, in the spirit of [KO06].

Introduction

Borrowing the words of [KOS06], simple Harnack curves are real algebraic curves sitting in toric surfaces “in the best possible way”. The original topological definition of [Mik00] can be rephrased in term of maximality with respect to a finite collection of Harnack-Smith inequalities, see [Mik01]. This maximality manifests in several interesting and different ways, as shown in [MR01], [PR11] or [MO07]. In particular, the reformulation of [MR01] has both advantages of being short and explicit : for a simple Harnack curve \mathcal{C} in

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a toric surface \mathcal{T}_Δ , the amoeba map $\mathcal{A} : \mathcal{C} \rightarrow \mathbb{R}^2$ is at most 2-to-1. The latter property characterizes simple Harnack curves if one allows them to have singularities. Indeed, it can be shown that any real oval of a simple Harnack curve can be contracted to a solitary double point (see [KO06]), and that the 2-to-1 property of the amoeba map \mathcal{A} is still satisfied for these singular simple Harnack curves (see [MR01]).

In the present paper, we introduce a generalization of the notion of simple Harnack curve, unifying the two classes of curves mentioned above. This generalization consists in relaxing the smoothness assumption in the characterization given in [PR11] : for a curve \mathcal{C} in a toric surface \mathcal{T}_Δ , its logarithmic Gauss map $\gamma : \mathcal{C} \dashrightarrow \mathbb{CP}^1$ is a rational map defined only on the smooth locus of \mathcal{C} . Its pullback to the normalization $\tilde{\mathcal{C}}$ of \mathcal{C} extends to an algebraic map $\tilde{\gamma} : \tilde{\mathcal{C}} \rightarrow \mathbb{CP}^1$.

Definition. *An irreducible real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a (generalized) simple Harnack curve if and only if its logarithmic Gauss map $\tilde{\gamma}$ is totally real, that is*

$$\tilde{\gamma}^{-1}(\mathbb{RP}^1) = \mathbb{RC}.$$

We then show that the characterization of [MO07] in terms of covering of the argument torus extends as well, implying that the area of the coamoeba of a simple Harnack curve is determined by its Euler characteristic.

In a second time, we address the question of construction of such curves. The existence of simple Harnack curves as introduced by Mikhalkin was proven using Viro's patchworking method. In the present case, the construction of singular curves requires an enhanced version of patchworking, see [Shu12]. Alternatively, we choose here to invoke Mikhalkin's approximation theorem for phase-tropical curves. We introduce the notion of tropical Harnack curves and show the following.

Theorem. *The approximation of a tropical Harnack curve of degree Δ is a simple Harnack curve in \mathcal{T}_Δ .*

As one can expect, the type of the singularities of the approximating curve is predicted by the singularities of the tropical Harnack curve. In particular, the latter theorem implies the existence of simple Harnack curves with a single hyperbolic node. We then classify all possible topological type of simple Harnack curves with a single hyperbolic node obtained by tropical approximation. By topological type of a curve $\mathcal{C} \subset \mathcal{T}_\Delta$, we mean the

topological triad

$$\left(\mathbb{R}\mathcal{T}_\Delta, \mathbb{R}\mathcal{C}, \bigcup_s \mathbb{R}\mathcal{D}_s\right)$$

where the \mathcal{D}_s 's are the toric divisors at infinity of the toric surface \mathcal{T}_Δ .

Proposition. *For a fixed degree Δ , the deformation classes of tropical Harnack curves with a single hyperbolic node are indexed by the smooth corners of Δ . The topological type of any algebraic approximation within a deformation class indexed by a corner ν is constant, and denoted $Top(\Delta, \nu)$.*

Here, smooth corners are corners of Δ corresponding to smooth points in the associated toric surface \mathcal{T}_Δ .

In the last part, we eventually undertake the topological classification of all simple Harnack curves with a single hyperbolic node, similarly to [Mik00]. We obtain a complete classification that shows that the algebraic picture is totally described by the tropical one.

Theorem. *For any simple Harnack curve \mathcal{C} in \mathcal{T}_Δ , there exists a smooth corner ν of Δ such that the topological type of \mathcal{C} is given by $Top(\Delta, \nu)$.*

This theorem is a manifestation of a deeper connection between simple Harnack curves and their tropical avatars. This connection has first been observed in [KO06]. Recall that the spine of an algebraic curve in $(\mathbb{C}^*)^2$ is a “canonical” tropical curve sitting inside its amoeba, see [PR04]. The following theorem is in some sense the converse of the first theorem stated above.

Theorem. *The consideration of the spine induces a local diffeomorphism between the spaces of algebraic and tropical Harnack curves with a single hyperbolic node.*

The latter results are promising for a general correspondence between algebraic and tropical Harnack curves. Such a correspondence deserves to be investigated deeper in a further work.

All the statements given above will be reformulated in a rigorous way throughout the paper.

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1 Prerequisites

1.1 Logarithmic geometry of planar curves

In this text, $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$ will denote an algebraic curve in the complex 2-torus. Such curve can be defined as the zero set of a Laurent polynomial

$$f \in \mathbb{C} [z^{\pm 1}, w^{\pm 1}] .$$

The coordinates of $(\mathbb{C}^*)^2$ induce a canonical isomorphism $z^\alpha w^\beta \mapsto (\alpha, \beta)$ between its space of characters and \mathbb{Z}^2 . The Newton polygon $New(f)$ is defined as the convex hull in $\mathbb{R}^2 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R}$ of the monomials $z^\alpha w^\beta$ appearing in f . As a convention, one will always consider polynomials $f \in \mathbb{C} [z, w]$ such that $New(f)$ touches both the z - and w - axes. According to this convention, any curve $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$ is defined by a polynomial f , unique up to a multiplicative constant $a \in \mathbb{C}^*$. The Newton polygon $New(f)$ is uniquely determined by

\mathcal{C}° and will be denoted Δ .

The Newton polygon Δ of \mathcal{C}° induces a toric compactification $(\mathbb{C}^*)^2 \subset \mathcal{T}_\Delta$, whenever the interior of Δ is 2-dimensional. The action of $(\mathbb{C}^*)^2$ onto itself extends to \mathcal{T}_Δ . Denote

$$S^1 := \{z \in \mathbb{C} \mid |z| = 1\} \simeq \mathbb{R}/2\pi\mathbb{Z}.$$

The moment map $\mu : \mathcal{T}_\Delta \rightarrow \Delta$ is the quotient map of \mathcal{T}_Δ by the action of $(S^1)^2$. To a side s of Δ , one associates the toric divisor at infinity associated $\mathcal{D}_s := \mu^{-1}(s)$. It is isomorphic to \mathbb{CP}^1 and compactifies one of the $(\mathbb{C}^*)^2$ -orbit. For any vertex v of Δ , $\mu^{-1}(v)$ is a fixed point of the $(\mathbb{C}^*)^2$ -action and will be referred to as the vertex of \mathcal{T}_Δ .

We will denote by $\mathcal{C} \subset \mathcal{T}_\Delta$ the closure of \mathcal{C}° . The curve \mathcal{C} intersects every divisor \mathcal{D}_s at $(|s \cap \mathbb{Z}^2| - 1)$ many points, counted with multiplicities. A curve $\mathcal{C} \subset \mathcal{T}_\Delta$ constructed as the closure of a curve $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$ of Newton polygon Δ never contains any vertex of \mathcal{T}_Δ . We will always restrict to this case while considering curves $\mathcal{C} \subset \mathcal{T}_\Delta$. We define the points at infinity of \mathcal{C} by $\mathcal{C}_\infty := \mathcal{C} \setminus \mathcal{C}^\circ$. We will also denote

$$b := |\partial\Delta \cap \mathbb{Z}^2| \quad \text{and} \quad g := |\text{Int } \Delta \cap \mathbb{Z}^2|.$$

The integer g is the arithmetic genus of \mathcal{C} and b is the intersection multiplicity of \mathcal{C} with the union of all the divisors at infinity, see for instance [Kho78] and [Kus76]. In particular, if \mathcal{C} intersects each of them transversally, then $|\mathcal{C}_\infty| = b$.

The moment map μ extends the amoeba map

$$\begin{aligned} \mathcal{A} : (\mathbb{C}^*)^2 &\rightarrow \mathbb{R}^2 \\ (z, w) &\mapsto (\log |z|, \log |w|) \end{aligned}$$

after a barycentric change of coordinates $\mathbb{R}^2 \rightarrow \text{Int } \Delta$. Denote also by $\arg(z) \in S^1$ the argument of the complex number $z \in \mathbb{C}^*$, the argument torus $T := (S^1)^2$ and

$$\begin{aligned} \text{Arg} : (\mathbb{C}^*)^2 &\rightarrow T \\ (z, w) &\mapsto (\arg(z), \arg(w)) \end{aligned} \quad .$$

Definition 1.1. Let $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$ be an algebraic curve. The amoeba (resp. coamoeba) of \mathcal{C}° is the subset $\mathcal{A}(\mathcal{C}^\circ) \subset \mathbb{R}^2$ (resp. $\text{Arg}(\mathcal{C}^\circ) \subset (S^1)^2$). With a slight abuse, we will as well refer to the latter subsets as $\mathcal{A}(\mathcal{C})$ and $\text{Arg}(\mathcal{C})$ respectively.

Proposition 1.2 (see [FPT00]). For an algebraic curve $\mathcal{C}^\circ \subset (\mathbb{C}^*)^2$, its amoeba $\mathcal{A}(\mathcal{C}^\circ)$ is a closed subset of \mathbb{R}^2 . Moreover, every connected component of $\mathbb{R}^2 \setminus \mathcal{A}(\mathcal{C}^\circ)$ is convex.

Amoebas and coamoebas are related by the fact that they are respectively real and imaginary part of algebraic curves in logarithmic coordinates. Interplays between them can be often described in term of the logarithmic Gauss map. For a smooth curve $\mathcal{C} \subset \mathcal{T}_\Delta$ given by a polynomial f , the logarithmic Gauss $\gamma : \mathcal{C} \rightarrow \mathbb{CP}^1$ is given on \mathcal{C}° by

$$\gamma(z, w) = [z \cdot \partial_z f(z, w) : w \cdot \partial_w f(z, w)]$$

This map is locally the composition of (any branch of) the coordinate wise complex logarithm with the classical Gauss map which associates to every point of a smooth hypersurface its tangent hyperplane. In the singular case, γ is only defined on the smooth part of \mathcal{C} . If $\pi : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$ is the normalization of \mathcal{C} , the removable singularity theorem allows to extend the rational map $\pi \circ \gamma$ to an algebraic map $\tilde{\gamma} : \tilde{\mathcal{C}} \rightarrow \mathbb{CP}^1$.

Definition 1.3. For a possibly singular curve $\mathcal{C} \subset \mathcal{T}_\Delta$ and normalization $\pi : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$, define the logarithmic Gauss map of $\tilde{\mathcal{C}}$ to be $\tilde{\gamma} : \tilde{\mathcal{C}} \rightarrow \mathbb{CP}^1$.

For a singular curve $\mathcal{C} \subset \mathcal{T}_\Delta$, we will always denote its normalization by $\pi : \tilde{\mathcal{C}} \rightarrow \mathcal{C}$. We will also denote $\tilde{\mathcal{C}}^\circ := \pi^{-1}(\mathcal{C}^\circ)$ and $\tilde{\mathcal{C}}_\infty := \tilde{\mathcal{C}} \setminus \tilde{\mathcal{C}}^\circ$. Remark that whenever \mathcal{C} has no singularity outside $(\mathbb{C}^*)^2$, \mathcal{C}_∞ and $\tilde{\mathcal{C}}_\infty$ are in bijection.

Let us give an alternative and useful description of γ (and then $\tilde{\gamma}$). Looking at z and w as 2 meromorphic functions on $\tilde{\mathcal{C}}$, consider the 2 meromorphic differentials $d \log(z)$ and $d \log(w)$ on \mathcal{C} . The quotient of two such differentials defines a meromorphic function on $\tilde{\mathcal{C}}$. One has the following

Proposition 1.4. The Logarithmic Gauss map $\tilde{\gamma} : \tilde{\mathcal{C}} \rightarrow \mathbb{CP}^1$ is given by

$$p \mapsto [-d \log(w(p)) : d \log(z(p))].$$

Moreover, the degree of $\tilde{\gamma}$ is $-\chi(\tilde{\mathcal{C}}^\circ)$.

Proof If p is a local coordinate on $\tilde{\mathcal{C}}^\circ$, then

$$[-d \log(w(p)) : d \log(z(p))] = \left[-\frac{d}{dp} \log(w(p)) : \frac{d}{dp} \log(z(p)) \right].$$

As $\tilde{\mathcal{C}}^\circ$ is immersed in $(\mathbb{C}^*)^2$, $(\frac{d}{dp} \log(z(p)), \frac{d}{dp} \log(w(p)))$ is a non zero tangent vector for the coordinatewise logarithm of \mathcal{C}° at the point corresponding to p . Hence, the tangent plane at p is given by the equation

$$-d \log(w(p)) \cdot u + d \log(z(p)) \cdot v = 0,$$

which proves the first part of the statement.

The degree of $\tilde{\gamma}$ can be computed as the number of zeroes of $d \log(w) \cdot u + d \log(z) \cdot v$ for a generic non zero vector (u, v) . By genericity, one can assume that $\tilde{\mathcal{C}}_\infty$ is exactly the set of poles of this differential and all of them are simple. By Riemann-Roch, such differential has degree $g(\tilde{\mathcal{C}}) - 2$. Hence, it has $g(\tilde{\mathcal{C}}) + |\tilde{\mathcal{C}}_\infty| - 2$ zeroes. \square

The map $\mathcal{A} : \tilde{\mathcal{C}}^\circ \rightarrow \mathbb{R}^2$ is a map between smooth surfaces. Following [Mik00], denote by $\tilde{F}^\circ \subset \tilde{\mathcal{C}}^\circ$ the critical locus of \mathcal{A} , that is the set of points where \mathcal{A} is not submersive. Denote by \tilde{F} its closure in $\tilde{\mathcal{C}}$.

Lemma 1.5.

$$\tilde{F} = \tilde{\gamma}^{-1}(\mathbb{RP}^1).$$

Moreover, \tilde{F} is also the closure of the critical locus of the map $\text{Arg} : \tilde{\mathcal{C}}^\circ \rightarrow T$.

Proof Let $\tilde{p} \in \tilde{\mathcal{C}}^\circ$ and denote $p := (p_1, p_2) := \pi(\tilde{p})$. The point \tilde{p} is in \tilde{F}° if $T_p \mathcal{C}^\circ$ contains a vector v tangent to the torus $|z| = |p_1|$, $|w| = |p_2|$ (if p is a singular point of \mathcal{C}° , consider the tangent line in $T_p \mathcal{C}^\circ$ corresponding to \tilde{p}). Equivalently, z has purely imaginary logarithmic coordinates. This holds if and only if $\tilde{\gamma}(\tilde{p}) \in \mathbb{RP}^1$.

Similarly, \tilde{p} is a critical point for the map Arg if $T_p \mathcal{C}^\circ$ contains a vector v tangent to $\arg(z) = \arg(p_1)$, $\arg(w) = \arg(p_2)$, i.e. z has real logarithmic coordinates. Once again, it holds if and only if $\tilde{\gamma}(\tilde{p}) \in \mathbb{RP}^1$. \square

Remark. Any point of $\tilde{\mathcal{C}}$ mapped to the boundary of $\mathcal{A}(\mathcal{C})$ belongs to \tilde{F} . By the above lemma, the real part $\mathbb{R}\tilde{\mathcal{C}}$ of any curve $\tilde{\mathcal{C}}$ defined over \mathbb{R} is always a subset of \tilde{F} .

Now let us recall that to any holomorphic function f on $(\mathbb{C}^*)^2$, one can associate its Ronkin function

$$N_f(x, y) := \frac{1}{(2i\pi)^2} \int_{\mathcal{A}^{-1}(x, y)} \frac{\log |f(z, w)|}{zw} dz \wedge dw$$

defined on \mathbb{R}^2 . The function N_f allows to describe the geometry of the amoeba $\mathcal{A}(\{f = 0\})$. It is a convex function that is affine linear on the connected components of the complement of $\mathcal{A}(\{f = 0\})$, see [PR04]. The gradient $\text{grad } N_f$ is then a constant function on such component. One defines the order of such component to be the value $\text{grad } N_f$ on it. In the case where f is a polynomial, $\text{grad } N_f$ takes values inside of $\text{New}(f)$. Moreover, its image is dense there.

Proposition 1.6 (see [FPT00]). *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be an algebraic curve. The order map defines an injection from the set of connected components of $\mathbb{R}^2 \setminus \mathcal{A}(\mathcal{C})$ to $\Delta \cap \mathbb{Z}^2$. Compact components are sent in the interior of Δ and non-compact components are sent on its boundary. Moreover, the order map is surjective on the vertices of Δ .*

Consideration of the Hessian of N_f gives rise to a so-called Monge-Ampère measure supported on $\mathcal{A}(\mathcal{C})$. Comparison of this measure with the standard Lebesgue measure gives the following result.

Proposition 1.7 (see [PR04]). *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be an algebraic curve. Then*

$$\text{Area}(\mathcal{A}(\mathcal{C})) \leq \pi^2 \text{Area}(\Delta)$$

where the area is computed with respect to the standard Lebesgue measure.

There are other “areas” one can compute about amoebas and coamoebas. Consider the complex orientation on the curve $\mathcal{C} \subset \mathcal{T}_\Delta$ and the counter clockwise orientation on \mathbb{R}^2 and $(S^1)^2$. For any non critical value p of $\mathcal{A}(\mathcal{C})$ (resp. $\text{Arg}(\mathcal{C})$), define the signed multiplicity of p to be the number of preimages with signs depending on the local change of orientation. Define also the positive multiplicity of p just to be the number of preimages.

Definition 1.8. Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be an algebraic curve. Define $\text{Area}_{\mathcal{A},s}(\mathcal{C})$ (resp. $\text{Area}_{\text{Arg},s}(\mathcal{C})$) to be the area of the amoeba (resp. the coamoeba) of \mathcal{C} counted with signed multiplicities, and $\text{Area}_{\mathcal{A},m}(\mathcal{C})$ (resp. $\text{Area}_{\text{Arg},m}(\mathcal{C})$) the area counted with positive multiplicities.

The following observation is due to Mikhalkin.

Lemma 1.9. For any algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$, the 2-forms $\mathcal{A}^*(dx_1 \wedge dx_2)$ and $\text{Arg}^*(dy_1 \wedge dy_2)$ are equal on $\tilde{\mathcal{C}}^\circ \setminus \tilde{F}^\circ$. It implies that

$$\text{Area}_{\mathcal{A},s}(\mathcal{C}) = \text{Area}_{\text{Arg},s}(\mathcal{C}) = 0$$

and

$$\text{Area}_{\mathcal{A},m}(\mathcal{C}) = \text{Area}_{\text{Arg},m}(\mathcal{C}).$$

Proof Consider locally the coordinate wise complex logarithm on $\tilde{\mathcal{C}}$. Its image is a holomorphic curve in \mathbb{C}^2 . It implies that the restriction of the complex 2-form $dz_1 \wedge dz_2$ on \mathbb{C}^2 to $\text{Log}(\tilde{\mathcal{C}})$ is zero. Write $z_j = x_j + iy_j$ for $j = 1, 2$, then $\text{Re}(dz_1 \wedge dz_2) = dx_1 \wedge dx_2 - dy_1 \wedge dy_2$ is also zero on $\text{Log}(\tilde{\mathcal{C}})$, meaning that the 2-forms $dx_1 \wedge dx_2$ and $dy_1 \wedge dy_2$ are equal on $\text{Log}(\tilde{\mathcal{C}})$. As we already said, the projection on the x -plane (resp. y -plane) is nothing but \mathcal{A} (resp. Arg). It implies the first part of the statement.

The equalities $\text{Area}_{\mathcal{A},s} = \text{Area}_{\text{Arg},s}$ and $\text{Area}_{\mathcal{A},m} = \text{Area}_{\text{Arg},m}$ are direct consequences. In the first case, the moment map μ extends \mathcal{A} and has a compact source space. Then it has a well defined degree d which is zero as μ is not surjective. This degree is precisely the number of preimages of \mathcal{A} counted with signs. Hence $\text{Area}_{\mathcal{A},s}(\mathcal{C}) = 0$. \square

Denote by A_f the set of connected components of the complement of $\mathcal{A}(\{f = 0\})$. For an element $\alpha \in A_f$ denote by N_f^α the affine linear function on \mathbb{R}^2 extending $(N_f)|_\alpha$. Then, the spine \mathcal{S}_f is defined as the corner locus of the piecewise affine linear and convex function

$$S_f := \max_{\alpha \in A_f} N_f^\alpha.$$

\mathcal{S}_f is a piecewise linear graph in the plane. Equipped with some natural collection of weights, the spine turns out to be a tropical curve, see next subsection.

Theorem 1.10 (see [PR04]). *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be an algebraic curve defined by a polynomial f . Then, $\mathcal{A}(\mathcal{C})$ deformation retracts on \mathcal{S}_f .*

1.2 Simple Harnack curves

Simple Harnack curves as considered here have been introduced in [Mik00]. Recall that a smooth real algebraic curve \mathcal{C} of genus g is said to be an M-curve if $\mathbb{R}\mathcal{C} \subset C$ has the maximal number of connected components, that is $g + 1$. Now, for a subset I of the set of sides of Δ , denote

$$\mathcal{C}_\infty^I := \mathcal{C} \cap \bigcap_{s \in I} \mathcal{D}_s \text{ and } \mathbb{R}\mathcal{C}_\infty^I := \mathbb{R}\mathcal{C} \cap \bigcap_{s \in I} \mathbb{R}\mathcal{D}_s.$$

Here is the original definition of [Mik00].

Definition 1.11. *A smooth real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack curve if*

- * \mathcal{C} is an M-curve,
- * *there exists a connected component $\alpha \subset \mathbb{R}\mathcal{C}$ and pairwise disjoint connected arcs $\alpha_\sigma \subset \alpha$ for every side σ of Δ such that $\mathcal{C}_\infty^\sigma \subset \alpha_\sigma$ and such that the cyclical ordering of the α_σ 's on α corresponds to the cyclical ordering of the σ 's on $\partial\Delta$.*

In [Mik00], Mikhalkin showed that the embedding of simple Harnack curves in toric surface depends only on the toric surface itself. Namely, one has

Theorem 1.12. *For any simple Harnack curve in \mathcal{T}_Δ , the topological triad*

$$\left(\mathbb{R}\mathcal{T}_\Delta, \mathbb{R}\mathcal{C}, \bigcup_s \mathbb{R}\mathcal{D}_s \right)$$

is unique and depends only on Δ . Here, the union runs over all the sides s of Δ .

Such theorem can be proved after a careful study of the amoeba of such a curve, showing that the latter is covered in a 2-to-1 fashion away from its boundary. It leads to the main theorem of [MR01]. In the latter, the authors showed that simple Harnack curves can only degenerate to singular curves with solitary double points, after contractions of real ovals. These curves are referred to as singular simple Harnack curves. They obtained the following.

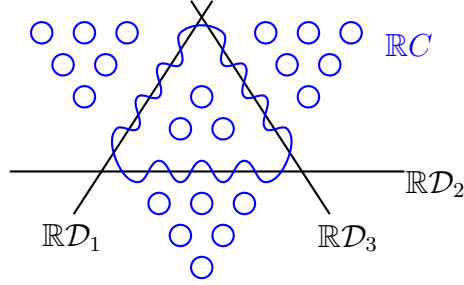


Figure 1: A simple Harnack curve of degree 8 in \mathbb{RP}^2 .

Theorem 1.13. *A real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a (possibly singular) simple Harnack curves if and only if one of the following conditions is satisfied*

- * $\mathcal{A} : \mathcal{C} \rightarrow \mathbb{R}^2$ is at most 2-to-1.
- * $\text{Area}(\mathcal{A}(\mathcal{C})) = \pi^2 \text{Area}(\Delta)$.

There are several others equivalent definitions for simple Harnack curves. The following is of particular interest for us, as it will be the starting point of the generalization suggested in this text.

Theorem 1.14 (see [Mik00] and [PR11]). *A smooth real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack curves if and only if its logarithmic Gauss map $\gamma : \mathcal{C} \rightarrow \mathbb{CP}^1$ is totally real, that is*

$$\gamma^{-1}(\mathbb{RP}^1) = \mathbb{RC}.$$

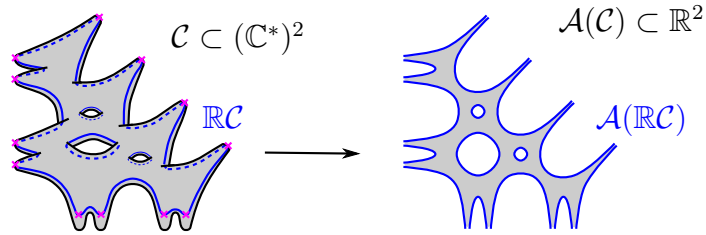


Figure 2: The amoeba map on a simple Harnack quartic.

The equivalent definition given in [MO07] will be generalized in theorem 1. The equivalent definition given in [Mik01] will also be discussed at the end of the section 3.1.

1.3 Phase-tropical curves

1.3.1 Tropical curves

Let us recall briefly some classical notions about tropical curves in the plane. All definitions, statements and their proofs can be found in [Mik05], [IMS09], and [BIMS15].

A tropical Laurent polynomial in two variables x and y is a function

$$f(x, y) = \text{“} \sum_{(\alpha, \beta) \in A} c_{(\alpha, \beta)} x^\alpha y^\beta \text{”}$$

where $A \subset \mathbb{Z}^2$ is a finite set and the usual arithmetic operations are replaced by the tropical ones

$$\text{“}x + y\text{”} = \max\{x, y\} \text{ and } \text{“}xy\text{”} = x + y.$$

Such a function is piecewise affine linear and convex. The Newton polygon $New(f)$ of f is the convex hull of A in $\mathbb{R}^2 = \mathbb{Z}^2 \otimes_{\mathbb{Z}} \mathbb{R}$. The tropical zero set $V(f)$ of a tropical Laurent polynomial f is defined as the subset of \mathbb{R}^2 where f is not smooth or equivalently, it is the set of points where at least two tropical monomials have the same value. As a first observation, a tropical zero set is a piecewise linear graph in \mathbb{R}^2 with rational slopes.

If g is another tropical Laurent polynomial given by

$$g(x, y) := \text{“}c_{(\alpha, \beta)} x^\alpha y^\beta \cdot f(x, y)\text{”},$$

then

$$V(f) = V(g),$$

but the converse fails to be true. Without loss of generality, we can and we do restrict once again to Newton polygons Δ contained in the positive quadrant and touching the two coordinate axes.

For a tropical polynomial f of Newton polygon Δ , consider its extended Newton polygon

$$\tilde{\Delta} := \text{ConvexHull} \{ ((\alpha, \beta), t) \in \mathbb{R}^3 \mid (\alpha, \beta) \in A, t \geq c_{(\alpha, \beta)} \}.$$

The projecting on the first two coordinates maps down all closed bounded faces of $\tilde{\Delta}$ homeomorphically onto Δ . It induces a subdivision Subdiv_f of Δ . Using the Legendre transform, one can observe the following duality

Proposition 1.15. *Let f be a tropical polynomial in two variables. The subdivision of \mathbb{R}^2 by $V(f)$ is dual to the subdivision Subdiv_f of Δ in the following sense*

- * 2-cells of $\mathbb{R}^2 \setminus V(f)$ are in bijection with vertices of Subdiv_f , and 2-cells of Subdiv_f are in bijection with vertices of $V(f)$,
- * leaves-edges of $V(f)$ are in bijection with edges of Subdiv_f , and their directions are orthogonal to each other,
- * incidence relations are reversed.

Moreover, unbounded 2-cells of $\mathbb{R}^2 \setminus V(f)$ are dual to boundary points of Δ and leaves of $V(f)$ are dual to edges on the boundary of Δ .

Definition 1.16. *Let f be a tropical polynomial in two variables. For any leaf-edge ε of $V(f)$, the weight $w(\varepsilon)$ of ε is the integer length of its dual edge ε^\vee in Subdiv_f , that is $|\varepsilon^\vee \cap \mathbb{Z}^2| - 1$.*

Definition 1.17. *A tropical curve $C \subset \mathbb{R}^2$ is a tropical zero set equipped with the weights defined in 1.16. If Δ is its Newton polygon, denote by Subdiv_C the subdivision of Δ dual to C .*

Remark. The convex piecewise affine linear function S_f defining the spine of the curve $\{f = 0\} \subset (\mathbb{C}^*)^2$ is a tropical polynomial. Equipped with the corresponding collection of weights, the spine of an algebraic curve in $(\mathbb{C}^*)^2$ is a tropical curve.

Among tropical curves, some of them will be of particular interest for us. They have both advantages of being very simple and generic.

Definition 1.18. *A tropical curve $C \subset \mathbb{R}^2$ is simple if its dual subdivision Subdiv_C contains solely triangles and parallelogram.*

In other words, a simple tropical curve has only 3-valent vertices and 4-valent vertices given locally as the union of two segments. Simple tropical curves can be parametrized uniquely by an abstract 3-valent tropical curve, see [Mik05]. Rather than defining tropical morphisms on abstract tropical curve, we give the following definition instead.

Definition 1.19. *The normalization \tilde{C} of a simple tropical curve $C \subset \mathbb{R}^2$ is the 3-valent graph obtained as the proper transform of C by the real blow-up of \mathbb{R}^2 at all its 4-valent vertices. Denote the blow-up by $\pi : \tilde{C} \rightarrow C$.*

Definition 1.20. *A simple tropical curve $C \subset \mathbb{R}^2$ is irreducible if its normalization \tilde{C} is connected.*

Definition 1.21. *From now on, we define the vertices (resp. edges, resp. leaves) of a simple tropical curve $C \subset \mathbb{R}^2$ to be the image of the vertices (resp. edges, resp. leaves) of its normalization \tilde{C} by the map π . They form the set $V(C)$ (resp. $E(C)$, resp. $L(C)$) and we denote $LE(C) := L(C) \cup E(C)$. As a convention, edges and leaves are always open.*

The points of C having two preimages in \tilde{C} are called the nodes of C and form the set $N(C)$.

Definition 1.22. *Let $C \subset \mathbb{R}^2$ be a simple tropical curve and $n \in N(C)$. The multiplicity of the node n is the natural number*

$$m(n) := 2 \cdot \text{Area}(n^\vee)$$

where n^\vee is the 2-cell dual to n in Subdiv_C . A node n is hyperbolic if $m(n) = 2$.

We end up this subsection by recalling what is the stable intersection multiplicity of two tropical curves C_1 and C_2 . When C_1 and C_2 intersect transversally away from their vertices, their intersection number is defined by

$$m(C_1, C_2) := \sum_{p \in C_1 \cap C_2} m_p(C_1, C_2),$$

where the multiplicity $m_p(C_1, C_2)$ at an intersection point p is given by twice the area the 2-cell dual to p in $\text{Subdiv}_{C_1 \cup C_2}$.

For any curves C_1 and C_2 , there is a dense open subset $\mathcal{O} \subset \mathbb{R}^2$ such that for any $\vec{v} \in \mathcal{O}$, C_1 and $C_2 + \vec{v}$ intersect transversally as above. By the balancing condition, one sees that $m(C_1, C_2 + \vec{v})$ does not depend on \vec{v} . One defines this number to be the stable intersection of C_1 and C_2 . For more details, see [RST05]

1.3.2 Phase-tropical curves

As in section 6 in [Mik05], consider the change of the holomorphic structure

$$H_t : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2 \\ (z, w) \mapsto \left(|z|^{\frac{1}{\log(t)}} \frac{z}{|z|}, |w|^{\frac{1}{\log(t)}} \frac{w}{|w|} \right).$$

Note that

$$\mathcal{A} \circ H_t = \frac{1}{\log(t)} \mathcal{A}.$$

When one denotes $\mathcal{L} := \{(z, w) \in (\mathbb{C}^*)^2 \mid z + w + 1 = 0\}$, the sequence of topological surfaces $\{H_t(\mathcal{L})\}_{t>1}$ converges in Hausdorff distance to the so-called phase tropical line L , when $t \rightarrow \infty$. Topologically, L is obtained as the gluing of three holomorphic annuli to the coamoeba of \mathcal{L} , as pictured in figure 3. Its amoeba $\mathcal{A}(L)$ is the classical tropical line Λ with vertex at the origin.

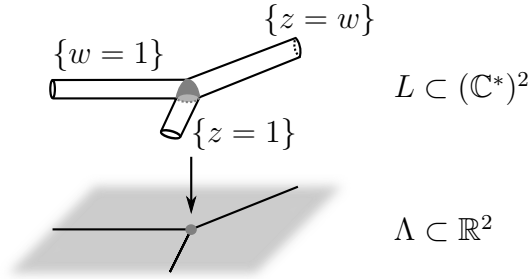


Figure 3: The fibration $\mathcal{A} : L \rightarrow \Lambda$.

Recall that a toric morphism $A : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2$ is a map of the form

$$(z, w) \mapsto (b_1 z^{a_{11}} w^{a_{12}}, b_2 z^{a_{21}} w^{a_{22}})$$

where $(b_1, b_2) \in (\mathbb{C}^*)^2$ and $(a_{ij}) \in M_{2 \times 2}(\mathbb{Z})$. It descends to an affine linear transformation on \mathbb{R}^2 (resp. on T) by composition with the projection \mathcal{A} (resp. Arg) that we still denote by A .

Definition 1.23. A general phase-tropical line $\Gamma \subset (\mathbb{C}^*)^2$ is the image of the phase-tropical line L by any toric morphism $A : (\mathbb{C}^*)^2 \rightarrow (\mathbb{C}^*)^2$. For a simple tropical curve $C \subset \mathbb{R}^2$ and any vertex $v \in V(C)$, denote by $T_v \subset C$ the tripod obtained as the union of v and its three adjacent leaves-edges in C .

Definition 1.24. A simple phase-tropical curve $V \subset (\mathbb{C}^*)^2$ is a topological surface such that :

- * its amoeba $\mathcal{A}(V)$ is a simple tropical curve $C \subset \mathbb{R}^2$,
- * for any $v \in V(C)$, there exists a general phase-tropical line $\Gamma_v \subset (\mathbb{C}^*)^2$ such that $(\mathcal{A}|_V)^{-1}(T_v) = \mathcal{A}^{-1}(T_v) \cap \Gamma_v$,
- * for any $e \in E(C)$, v_1 and v_2 its two adjacent vertices in C , Γ_{v_1} and Γ_{v_2} coincide on $\mathcal{A}^{-1}(e)$.

Remark. Phase-tropical curves have been introduced in [Mik05], with a slightly different terminology, and are extensively studied in the unpublished work [Mik]. For more details, one refers to [Lan15].

Similarly to Riemann surfaces, simple phase-tropical curves can be described in terms of Fenchel-Nielsen coordinates, see [Lan15]. Recall that the coamoeba $Arg(\mathcal{L})$ is the union of two open triangles delimited by three geodesics plus their three common vertices. As a convention, we fix a framing on each of the boundary geodesics of $Arg(\mathcal{L})$ according to the up-leftward triangle, see figure 4. These framings are fixed by any of the 6 toric automorphisms of \mathcal{L} .

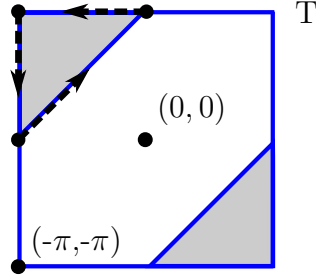


Figure 4: The coamoeba of \mathcal{L} , and the framings of its 3 boundary geodesics.

Now, recall that for any general phase-tropical line $\Gamma := A(L)$, one has that $\mathcal{A}(\Gamma) = A(\Lambda)$ and that the fiber in Γ over $\mathcal{A}(\Gamma)$ is $Arg(\Gamma) = A(Arg(\Lambda)) = A(Arg(\mathcal{L}))$. For any general phase-tropical line $\Gamma := A(L)$, one carries the framings of $Arg(\Lambda)$ to framings on $Arg(\Gamma) = A(Arg(\Lambda))$ by the map A . By construction, it does not depend on the choice of A .

With these framings, any boundary geodesic is canonically isomorphic as an abelian group to $S^1 \subset \mathbb{C}$.

For a simple phase-tropical curve V , with $C := \mathcal{A}(V)$, and two vertices $v_1, v_2 \in V(C)$ connected by an edge $e \in E(C)$, the holomorphic annulus $(\mathcal{A}|_V)^{-1}(e)$ maps to a geodesic γ_e in the argument torus. This is a common boundary geodesic of the two coamoebas $(\mathcal{A}|_V)^{-1}(v_1)$ and $(\mathcal{A}|_V)^{-1}(v_2)$. The two induced framings on γ_e define an orientation reversing isometry of the form

$$\begin{aligned} \tau_2^{-1} \circ \tau_1 : S^1 &\rightarrow S^1 \\ z &\mapsto -\overline{e^{i\theta} z}. \end{aligned}$$

Definition 1.25. For any $e \in E(C)$, the element $e^{i\theta} \in S^1$ constructed above is called the twist parameter of the edge e .

Definition 1.26. A simple real-tropical curve $V \subset (\mathbb{C}^*)^2$ is a simple phase-tropical curve that is invariant under complex conjugation. We denote by $\mathbb{R}V \subset (\mathbb{R}^*)^2$ its real point set and by $\mathbb{T}V$ the image of $\mathbb{R}V$ under the diffeomorphism

$$\begin{aligned} \mathcal{A}_s : (\mathbb{R}^*)^2 &\rightarrow \mathbb{R}^2 \times (\mathbb{Z}_2)^2 \\ (x, y) &\mapsto \left(\left(\frac{x}{|x|} \ln |x|, \frac{y}{|y|} \ln |y| \right), \left(\frac{x}{|x|}, \frac{y}{|y|} \right) \right). \end{aligned}$$

Remark. Note that by composing the map \mathcal{A}_s with

$$\begin{aligned} Abs : \mathbb{R}^2 \times (\mathbb{Z}_2)^2 &\rightarrow \mathbb{R}^2 \\ ((x, y), (\varepsilon, \delta)) &\mapsto (\varepsilon x, \delta y), \end{aligned}$$

one recovers the map \mathcal{A} . Note moreover that the real locus of any holomorphic annulus $(\mathcal{A}|_V)^{-1}(e)$ above $e \in LE(C)$ has exactly two connected components, whenever it is defined over \mathbb{R} .

The following results are easy consequences of the above definitions. The proofs are left to the reader.

Proposition 1.27. Let $V \subset (\mathbb{C}^*)^2$ be a simple real tropical curve and denote by $C := \mathcal{A}(V)$ its amoeba. Then $\mathbb{T}V$ is a piecewise linear curve and $Abs : \mathbb{T}V \rightarrow C$ induces a 2-to-1 correspondence between $LE(\mathbb{T}V)$ and $LE(C)$.

Proposition 1.28. The twist parameters of a real-tropical curve $V \subset (\mathbb{C}^*)^2$ are always contained in $\{-1, 1\} \subset S^1$.

2 Definition and construction

2.1 Definition and first properties

From now on, the notion of simple Harnack curve will always refer to the following definition.

Definition 2.1. *An irreducible real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack curve if it is irreducible and*

$$\tilde{F} = \mathbb{R}\tilde{\mathcal{C}}.$$

By the lemma 1.5, one has the following reformulation.

Definition 2.2. *An irreducible real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack curve if and only if its logarithmic Gauss map $\tilde{\gamma}$ is totally real.*

This generalizes the notion of simple Harnack curve originally given by Mikhalkin, via the theorem 1.14. The only difference here is that a simple Harnack curve is not required to be smooth. We will see later that it allows many other cases to appear.

Remark. The latter definition of simple Harnack curve implies that $\mathbb{R}\tilde{\mathcal{C}} \subset \tilde{\mathcal{C}}$ is of type 1, i.e $\mathbb{R}\tilde{\mathcal{C}} \setminus \tilde{\mathcal{C}}$ has exactly two connected components.

In order to get the first general properties of such curves, let us reproduce a construction due to [MO07] : for a curve $\mathcal{C} \subset \mathcal{T}_\Delta$ consider the map

$$\begin{aligned} \text{Alga} : \mathcal{C}^\circ &\rightarrow T \\ (z, w) &\mapsto (2 \arg(z), 2 \arg(w)). \end{aligned}$$

Define $\tilde{\mathcal{C}}_{Bl}$ to be the real blow-up of $\tilde{\mathcal{C}}$ at every point of $\tilde{\mathcal{C}}_\infty$. For any point $p \in \tilde{\mathcal{C}}_\infty$ denote by \mathbb{P}_p the fiber of $\tilde{\mathcal{C}}_{Bl} \rightarrow \tilde{\mathcal{C}}$ over p . By construction, one has the factorisation $\tilde{\mathcal{C}}_{Arg} \rightarrow \tilde{\mathcal{C}}_{Bl} \rightarrow \tilde{\mathcal{C}}$ inducing a double covering $S_p \rightarrow \mathbb{P}_p$ for any $p \in \tilde{\mathcal{C}}_\infty$.

Lemma 2.3. *The map Alga naturally extends to*

$$\text{Alga} : \tilde{\mathcal{C}}_{Bl} \rightarrow T.$$

Proof By lemma 4.3, $Alga$ extend to $\tilde{\mathcal{C}}_{Arg}$. For any p and any point in \mathbb{P}_p , its two preimages in S_p are mapped to the same value by $Alga$. Hence, $Alga : \tilde{\mathcal{C}}_{Arg} \rightarrow T$ factorizes through $\tilde{\mathcal{C}}_{Bl}$. \square
Define the subset $\tilde{\mathcal{C}}_0 \subset \tilde{\mathcal{C}}_{Bl}$ to be $Alga|_{\tilde{\mathcal{C}}_{Bl}}^{-1}(\{0_T\})$. Note that

$$Alga^{-1}(\{0_T\}) = (\mathbb{R}^*)^2.$$

Thus, it implies that $\tilde{\mathcal{C}}_0$ is the union of $\mathbb{R}\tilde{\mathcal{C}}_{Bl}$ plus some isolated points, whenever \mathcal{C} is defined over \mathbb{R} . In this case, the isolated points of $\tilde{\mathcal{C}}_0$ come either as the trace of solitary double points of $\mathbb{R}\mathcal{C}$ or from non transverse intersection with a toric divisor at infinity. Indeed, by lemma 4.3, if \mathcal{C} intersects a divisor \mathcal{D}_s with multiplicity m at a point $p \in \mathcal{C}_\infty$, there are exactly m points in the exceptional divisor \mathbb{P}_p belonging to $\tilde{\mathcal{C}}_0$, and exactly one of these belongs to $\mathbb{R}\tilde{\mathcal{C}}_{Bl}$.

Lemma 2.4. *An irreducible real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack curve if and only if*

$$Alga : \tilde{\mathcal{C}}_{Bl} \setminus \tilde{\mathcal{C}}_0 \rightarrow T \setminus \{0_T\}$$

is an unbranched covering.

Proof By lemma 1.5 and the remark above, the latter statement is an equivalent reformulation of definition 2.1. \square

Define at last \hat{T} to be the real blow-up of T at 0_T , and $\hat{\mathcal{C}}$ to be the real blow-up of $\tilde{\mathcal{C}}_{Bl}$ at $\tilde{\mathcal{C}}_0$. As blowing-up at a smooth submanifold of codimension 1 doesn't change the surface, blowing-up is effective only at isolated points of $\tilde{\mathcal{C}}_0$.

Theorem 1. *An irreducible real algebraic curve $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack curve if and only if the map $Alga : \tilde{\mathcal{C}}_{Bl} \setminus \tilde{\mathcal{C}}_0 \rightarrow T \setminus \{0_T\}$ extends to a covering*

$$Alga : \hat{\mathcal{C}} \rightarrow \hat{T}.$$

Proof Let \mathcal{C} be a simple Harnack curve. By definition, the map $Alga : \tilde{\mathcal{C}}_{Bl} \rightarrow T$ is regular at the isolated points of $\tilde{\mathcal{C}}_0$. Hence, the map $Alga : \tilde{\mathcal{C}}_{Bl} \setminus \tilde{\mathcal{C}}_0 \rightarrow T \setminus \{0_T\}$ extends to \hat{T} in a tautological way at these isolated points. At a point of $\mathbb{R}\tilde{\mathcal{C}}_{Bl}$, blowing-up consists of considering the

normal direction to $\mathbb{R}\tilde{\mathcal{C}}_{Bl}$ in the tangent space. This actually specifies the line bundle \mathcal{L}_{Im} introduced in the proof of lemma 4.2. The projectivized tangent map of $Alga$ realizes a covering of the exceptional divisor of \hat{T} by \mathcal{L}_{Im} , giving the extension $Alga : \hat{\mathcal{C}} \rightarrow \hat{T}$. Conversely, if \mathcal{C} is such that $Alga : \hat{\mathcal{C}} \rightarrow \hat{T}$, lemma 2.4 implies that \mathcal{C} is a simple Harnack curve. \square

Corollary 2.5. *If $\mathcal{C} \subset \mathcal{T}_\Delta$ is a simple Harnack curve, then*

$$Area_{Arg,m}(\mathcal{C}) = \pi^2(-\chi(\hat{\mathcal{C}})).$$

If moreover \mathcal{C} has no real solitary double points and if it intersecting transversally every toric divisors at infinity, then

$$Area_{Arg,m}(\mathcal{C}) = \pi^2(-\chi(\tilde{\mathcal{C}}^\circ)).$$

Proof It is clear by definition that $Area_{Arg,m}(\mathcal{C}) = 4 Area_{Alga,m}(\mathcal{C})$, where $Area_{Alga,m}(\mathcal{C})$ is defined similarly to $Area_{Arg,m}(\mathcal{C})$. By theorem 1,

$$Area_{Alga,m}(\mathcal{C}) = Area(T) \cdot \deg Alga = 4\pi^2 \cdot (-\chi(\hat{\mathcal{C}})).$$

The first part of the statement follows. For the second one, the assumptions are such that $\tilde{\mathcal{C}}_0$ has no solitary double point. It implies that $\hat{\mathcal{C}} = \tilde{\mathcal{C}}_{Bl}$, but $\chi(\tilde{\mathcal{C}}_{Bl}) = \chi(\tilde{\mathcal{C}}^\circ)$. \square

2.2 Tropical Harnack curves

Definition 2.6. *Let $C \subset \mathbb{R}^2$ be a simple tropical curve with normalization \tilde{C} . For every oriented loop $\tilde{\lambda} \subset \tilde{C}$, and λ the corresponding oriented loop in C , denote by $\Gamma_\lambda \subset E(C) \cap \lambda$ the subset of oriented edges that forms, together with its previous and following edges in λ , a non convex piecewise linear curve in \mathbb{R}^2 .*

Definition 2.7. *An irreducible simple tropical curve $C \subset \mathbb{R}^2$ with normalization \tilde{C} is a tropical Harnack curve if for every oriented loop $\tilde{\lambda} \subset \tilde{C}$, one has*

$$\sum_{\vec{e} \in \Gamma_\lambda} w(e) \cdot v_{\vec{e}} = 0 \mod 2 \quad (1)$$

where $v_{\vec{e}}$ is the primitive integer vector supporting the oriented edge \vec{e} , and $w(e)$ is the weight of e .

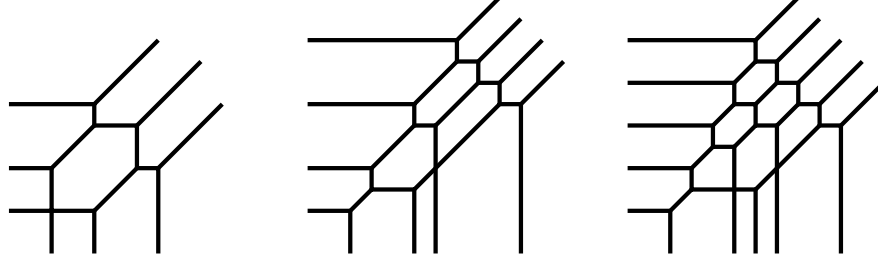


Figure 5: Tropical cubic, quadric and quintic. Both the cubic and the quintic are Harnack whereas the quartic does not satisfy (1) in definition 2.7.

Note the similarity with the condition of “twist-admissibility” of definition 3.3 in [BIMS15]. We now give another equivalent characterization of tropical Harnack curves.

Definition 2.8. *An edge (resp. a leaf) of $\mathbb{T}V$ is defined to be one of the two connected components mapping onto an edge (resp. a leaf) of C . They form a set denoted $E(\mathbb{T}V)$ (resp. $L(\mathbb{T}V)$). Their union is denoted $LE(\mathbb{T}V)$. Let $V \subset (\mathbb{C}^*)^2$ be a real-tropical curve. An inflection pattern of $\mathbb{T}V$ is a collection of three consecutive elements of $LE(\mathbb{T}V)$ that is not convex.*

Proposition 2.9. *An irreducible simple tropical curve $C \subset \mathbb{R}^2$ is a tropical Harnack curve if and only if there exists a simple real-tropical curve $V \subset (\mathbb{C}^*)^2$ such that*

- * $\mathcal{A}(V) = C$,
- * $\mathbb{T}V$ has no inflection pattern.

In such case, V is unique up to the four sign changes of the coordinates. We refer to such V as a phase-tropical Harnack curve.

Let us now recall how one can recover the curve $\mathbb{T}V$ of a phase-tropical Harnack curve from its underlying tropical Harnack curve $C := \mathcal{A}(V)$. According to propositions 1.27 and 2.9, each connected component of $\mathbb{T}V$ is convex and the map $Abs : LE(\mathbb{T}V) \rightarrow LE(C)$ is 2-to-1. It follows that each such connected components maps to exactly one of the boundary components of an infinitely thin ribbon R on the normalization of C (see figure 6). Hence, $\mathbb{T}V$ can be recovered from the data of a pair of signs on each connected

component of ∂R .

One has to make a choice as there are four possible phase-tropical curves V sitting above C . Once we fix the quadrant of one of the components of ∂R , the others are prescribed by the following rule. The signs of two components of ∂R that bound the same element of $e \in LE(C)$ are obtained by adding $(a, b) \bmod 2$, where $(-b, a)$ is a primitive integer vector supporting e . Indeed, the fiber of \mathcal{A}_V over e is a holomorphic annulus given by an equation of the form

$$z^{-b}w^a = c$$

where $c \in \mathbb{R}^*$. This annulus realizes the class $(a, b) \in H_1((\mathbb{C}^*)^2, \mathbb{Z})$ and maps to a geodesic of slope (a, b) in T under Arg . This geodesic contains exactly two points of the real subtorus $(\mathbb{Z}_2)^2 \subset T$ obtained one from the other by adding $(a, b) \bmod 2$.

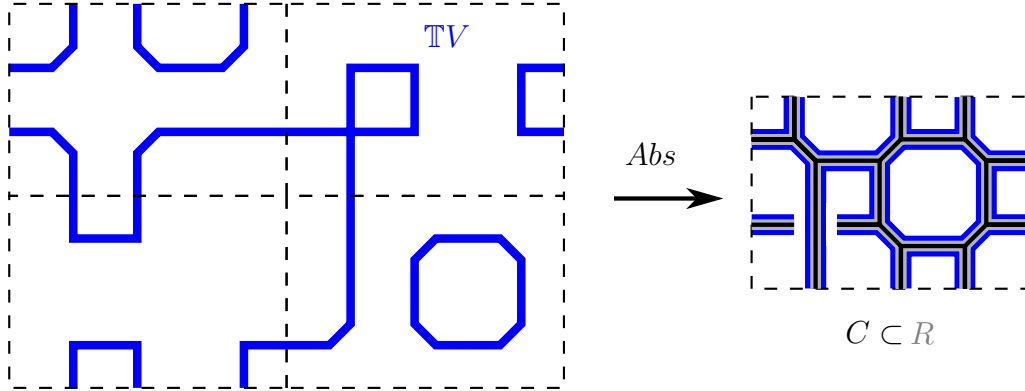


Figure 6: Recovering TV from the topological Harnack curve C .

We end up this section with the proof of proposition 2.9.

Definition 2.10. *The edge of a simple real-tropical curve is twisted (or has a twist) if its twist parameter is -1. It is not twisted (or has no twist) otherwise.*

Note that we made a slight abuse of language by speaking about edge of a simple phase-tropical curve rather than edge of its underlying simple tropical curve.

Lemma 2.11. *Let $V \subset (\mathbb{C}^*)^2$ be simple real-tropical curve. Then the inflection patterns of \mathbb{TV} are in 2-to-1 correspondence with the twisted edges of V .*

Proof The latter can be formulated in terms of signs distribution in combinatorial patchworking, see for example section 3 of [BIMS15] and references therein. A simple computation allows to describe the two possible cases pictured in figure 7 below. \square

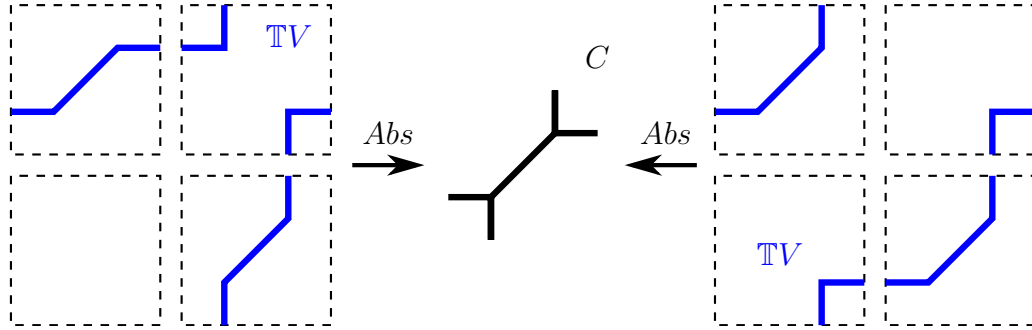


Figure 7: \mathbb{TV} over C in the twisted (left) and non-twisted case (right).

Proof of proposition 2.9 Suppose there is a simple phase-tropical curve $V \subset (\mathbb{C}^*)^2$ such that $\mathcal{A}(V) = C$ and \mathbb{TV} has no inflection pattern. By the previous lemma, it is equivalent to the fact that V has no twisted edges. For any vertex $v \in \lambda$, there is a distinguished point among the three special points of the coamoeba $(\mathcal{A}|_V)^{-1}(v)$, namely the intersection point of the two geodesics corresponding to the two edges in λ adjacent to v . Let us look at the position of this distinguished point in the argument torus T while going around λ . Going from a vertex v to the next one via an edge e , the point is moved according to the following rule : if e is not in Γ_λ , then this point is fixed; if e is in Γ_λ , this point is moved by $\pi \cdot w(e) \cdot v_{\vec{e}}$ in T . After a full cycle, the distinguished point has to come back to its initial place. This is clearly equivalent to the condition stated in definition 2.7 on the loop λ . Hence C is a tropical Harnack curve.

Conversely, if C satisfies the condition of definition 2.7 for any cycle, pick an initial vertex v_0 on C . There is exactly one possible general phase-tropical line Γ_{v_0} such that $(\mathcal{A}|_V)^{-1}(T_{v_0}) = \Gamma_{v_0} \cap \mathcal{A}^{-1}(T_{v_0})$, up to the four changes of signs of the coordinates. The twists determine the gluing of the general

phase-tropical lines above adjacent vertices along the common edge. Hence, once Γ_{v_0} is fixed, the adjacent general phase-tropical lines are also fixed. The first part of the proof shows that the condition of definition 2.7 is necessary and sufficient for this construction to close up along every cycle λ . The proposition is proved. \square

2.3 Construction by tropical approximation

After Viro introduced his patchworking techniques in the late 70's, see [Vir80], it has been extensively used to construct real algebraic hypersurfaces with prescribed topology. In the case of curves, Mikhalkin's approximation theorem even extends the possibilities given by Viro's method. In particular, the point of view of parametrized objects suits better for constructing singular curves. Here is a particular case of Mikhalkin's theorem.

Theorem 2.12 (Mikhalkin). *Let $V \subset (\mathbb{C}^*)^2$ be a simple real-tropical curve of Newton polygon Δ such that its normalization \tilde{V} has genus g and n punctures. Then, there exists a family of Riemann surfaces $\{S_t\}_{t>1} \subset \mathcal{M}_{g,n}$ together with a family of immersions $\iota_t : S_t \rightarrow (\mathbb{C}^*)^2$ such that*

- * $\iota(S_t)$ is a real algebraic curve of newton polygon Δ ,
- * $\iota(S_t)$ converges in Hausdorff distance to V .

Proof See [Lan15]. \square

Definition 2.13. *Let C be a tropical Harnack curve of Newton polygon Δ . Define $\text{Top}(C)$ to be the topological triad*

$$\left(\mathbb{RT}_\Delta, \mathbb{RV}, \bigcup_s \mathbb{RD}_s \right)$$

up to homeomorphism. Here, s runs over all the sides of Δ and V is one of the four simple phase-tropical curves of proposition 2.9 sitting above C .

Thanks to Mikhalkin's approximation theorem, one can prove the following.

Theorem 2. *Let C be a tropical Harnack curve of Newton polygon Δ , then there exists a simple Harnack curve $\mathcal{C} \subset \mathcal{T}_\Delta$ such that*

$$\left(\mathbb{R}\mathcal{T}_\Delta, \mathbb{R}\mathcal{C}, \bigcup_s \mathbb{R}\mathcal{D}_s \right) = \text{Top}(C).$$

Before giving the proof, let us briefly illustrate why the latter theorem provides new instances of simple Harnack curves. In the previous literature, simple Harnack curves have been considered with only real solitary double points as possible singularities, see [MR01] and [KO06] for example. By approximating tropical Harnack curves one can construct simple Harnack curve with hyperbolic nodes, see figure 6 for example.

As another new instance, One can construct simple Harnack curves with complex conjugated double points. In the figure 8, we illustrate the construction of a curve of bi-degree $(4, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Such curve has arithmetic genus 3. The curve pictured here has one hyperbolic node on its real part and the tropical node of multiplicity 4 is responsible for two complex-conjugated double points.

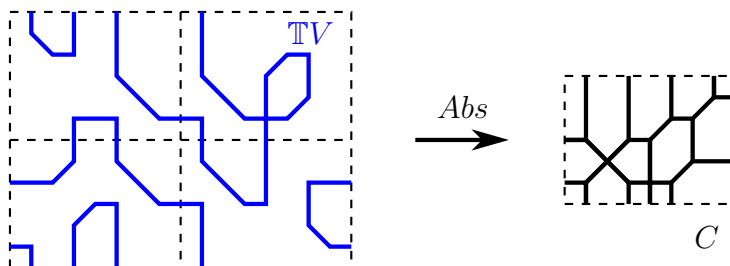


Figure 8: $\text{TV} \subset \mathbb{P}^1 \times \mathbb{P}^1$ for the tropical Harnack curve C .

We end up this section with the proof of theorem 2. Let V be a phase-tropical Harnack curve with $C := \mathcal{A}(V)$. By blowing up V , one obtains a smooth topological surface \tilde{V} with a real structure. It provides a parametrization $\iota : \mathbb{R}\tilde{V} \rightarrow \mathbb{R}V$. As $\mathbb{R}\tilde{V} \subset \tilde{V}$ is of type 1, the choice of an half of V defines an orientation on $\mathbb{R}\tilde{V}$. Now, any small deformation ι_ε of ι has a well defined logarithmic Gauss map γ_ε as long as $\mathcal{A} \circ \iota_\varepsilon$ is smooth. Orient $\mathbb{R}\tilde{V}$ such that $\deg \gamma_\varepsilon \geq 0$. This degree does not depend on the deformation ι_ε .

Definition 2.14. *Define the logarithmic curvature of $\mathbb{R}V$ to be $\deg \gamma_\varepsilon$.*

Proposition 2.15. *The logarithmic curvature of $\mathbb{R}V$ is equal to $-\chi(\tilde{V})$.*

Proof On one hand, $-\chi(\tilde{V})$ is equal to the number of vertices of \tilde{C} . To see this, cut \tilde{C} at the middle of any of its edges. As \tilde{C} has only 3-valent vertices, it splits into a collection of tripods, one for each vertex of \tilde{C} . The part of \tilde{V} sitting above any of these tripods is a topological pair-of-pants, having Euler characteristic -1. Additivity of Euler characteristic implies the above claim. On the other hand, one can compute the total logarithmic curvature of $\mathbb{R}V$ by computing its local contributions. By the very definition, these contributions are concentrated at the vertices of $\mathbb{T}V$. It corresponds to $1/\pi$ times the measure of the solid angle between the two normals at the vertices of $\mathbb{T}V$ (see figure 4 in [BLR13]), counted with signs depending on how $\mathbb{T}V$ changes inflection between to consecutive vertices. As $\mathbb{T}V$ has no inflection pattern, this local contributions should all be counted positively. Now, for each vertex of C , there correspond three vertices of $\mathbb{T}V$. One easily sees that the local contributions at these three vertices add up to 1, see figure 5 in [BLR13]. Hence the total logarithmic curvature of $\mathbb{R}V$ is also equal to the number of vertices of C , that is the number of vertices of \tilde{C} by convention. The result follows. \square

Proof of theorem 2. For t large enough, the immersed curve $\iota(S_t)$ in theorem 2.12 realizes the topological type $Top(C)$. Denoting by \mathcal{C} such immersed curve, it remains to show that, necessarily, \mathcal{C} is a simple Harnack curve. By the very definition of 2.14, one has that the degree of $\tilde{\gamma}|_{\mathbb{R}\mathcal{C}}$ is equal to the total logarithmic curvature of $\mathbb{R}V$, for any phase-tropical Harnack curve V sitting above C . This is equal to $-\chi(\tilde{V})$ by proposition 2.15. But $-\chi(\tilde{V}) = -\chi(\tilde{C}^\circ)$, by construction of \mathcal{C} , see theorem 2.12. By proposition 1.4, this is exactly the degree of $\tilde{\gamma}$. Hence, the logarithmic Gauss map $\tilde{\gamma}$ is totally real. By proposition 2.2, it implies that \mathcal{C} is a simple Harnack curve. \square

2.4 Tropical Harnack curves with a single hyperbolic node

The purpose of this section is to show that the topological types for tropical Harnack curves with a single hyperbolic node are indexed by the pairs (Δ, ν) where Δ is the degree of the curve and ν is a smooth vertex of Δ . Recall that a vertex is said to be smooth if the toric surface \mathcal{T}_Δ is smooth at the corresponding vertex.

Proposition 2.16. *Let $C \subset \mathbb{R}^2$ be a tropical Harnack curve of Newton polygon Δ with a single hyperbolic node n . Then the parallelogram n^\vee dual to n in Subdiv_C has exactly three of its vertices on the boundary of Δ . These three vertices are distributed on two sides of Δ that are adjacent to a smooth vertex ν of Δ .*

Proof Suppose n^\vee has at least two vertices v_1 and v_2 in the interior of Δ . Consider the polygonal domain P of Δ obtained by taking the union of n^\vee together with all the minimal triangle of Subdiv_C having v_1 or v_2 as a vertex. Consider the subset of C dual to P . Then, its normalization in \tilde{C} a single loop $\tilde{\lambda}$. There are two cases : either v_1 and v_2 are consecutive or opposite in n^\vee . In the first case, Γ_λ is a singleton. In the second, Γ_λ is exactly composed of the two leaves-edges forming the node n . In both case, the condition of definition 2.7 is not fulfilled. Hence, we get a contradiction.

If n^\vee has its four vertices on the boundary Δ , then C is reducible. This is a contradiction.

Then, n^\vee exactly three of its vertices on the boundary of Δ . Either the middle vertex is a vertex of Δ , and this is then a smooth one, or one of the two sides of n^\vee specified by the three vertices is on a side of Δ and a minimal triangle is attached to the other side. The vertex of this triangle not contained in n^\vee is the smooth vertex of Δ we are looking for. \square

Two projectively equivalent tropical polynomial of degree Δ give the same tropical curves, but the converse is not true in general. It only holds on the subspace of polynomials for which every tropical monomial dominate the others for at least one point. This is easily seen to be a closed polyhedral domain in \mathbb{RP}^{k-1} , where k is the number of integer point in Δ . Tropical curves of degree Δ will always be seen as point in this polyhedral domain.

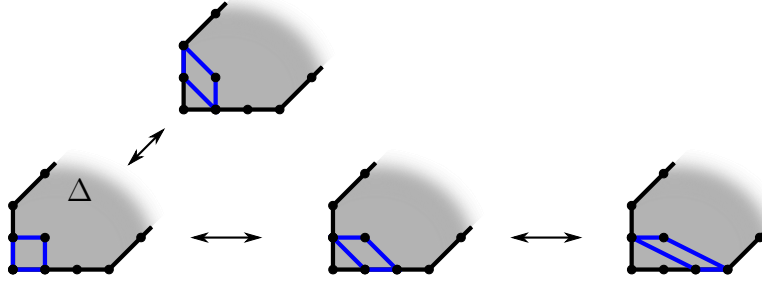
Definition 2.17. *Let $C \subset \mathbb{R}^2$ be a tropical Harnack curve with a single hyperbolic node n . We say that the node of C is next to ν if ν is the smooth vertex of the Newton polygon of C given in the previous proposition.*

Denote by $\mathcal{TH}_{\Delta,\nu}$ the interior of the closure of the space of tropical Harnack curve with a single hyperbolic node next to ν inside the space of tropical curves of degree Δ .

Remark. Curves of $\mathcal{TH}_{\Delta,\nu}$ are tropical curves with maximal number of compact holes and a 4-valent vertex as prescribed by proposition 2.16. The other vertices can be of any valency (greater than 3).

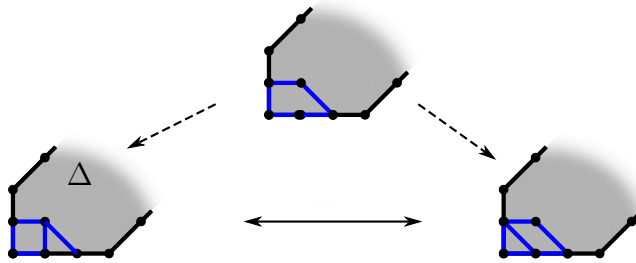
Proposition 2.18. *The topological type $Top(C)$ of any simple Harnack curve $C \in \mathbb{TH}_{\Delta,\nu}$ is unique and will be denoted by $Top(\Delta, \nu)$.*

Proof Up to toric transformation of $(\mathbb{C}^*)^2$, one can assume that $\nu = (0, 0)$ and that its two adjacent sides are supported on the x - and the y -axes. Consider the subdivision $Subdiv_C$ dual to Δ . By proposition 2.16, the unique parallelogram of $Subdiv_C$ can only be of one of the following type



The connected components of the space of tropical Harnack curves inside $\mathbb{TH}_{\Delta,\nu}$ are characterized by the combinatorial type of $Subdiv_C$. For a fixed position of the unique parallelogram of $Subdiv_C$, one can pass from one combinatorial type to another by allowing extra quadrilaterals. Doing so, one see that the unfolding procedure giving TV from C (see end of section 2.2) gives the same topological type.

In order to change the parallelogram of $Subdiv_C$ according to an arrow of the above picture, one need first to move to a particular combinatorial type. Indeed, $Subdiv_C$ has to contains the minimal triangle such that its union with the parallelogram is the trapezium obtained as the union of the two parallelograms at each sides of the concerned arrow, see the picture below.



Once again, we see that passing from one parallelogram to another by such

deformation does not change $Top(C)$. □

3 Simple Harnack curves with a single hyperbolic node

3.1 Statements of the main theorems

In this section we undertake the classification of the topological triads

$$\left(\mathbb{R}\mathcal{T}_\Delta, \mathbb{R}\mathcal{C}, \bigcup_s \mathbb{R}\mathcal{D}_s \right)$$

for simple Harnack curves \mathcal{C} with a single hyperbolic node in any toric surface \mathcal{T}_Δ . We obtain the following.

Theorem 3. *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve with a single hyperbolic node. Assume moreover that \mathcal{C} intersects transversally every toric divisor at infinity. Then, there is a smooth vertex ν of Δ such that*

$$\left(\mathbb{R}\mathcal{T}_\Delta, \mathbb{R}\mathcal{C}, \bigcup_s \mathbb{R}\mathcal{D}_s \right) = Top(\Delta, \nu).$$

In the latter theorem, one had to specify the intersection profil at infinity, as the topological classification depends on it. Transversality is a genericity assumption, and the general case can be deduced easily from the generic one.

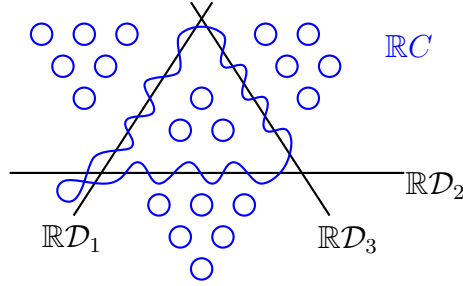


Figure 9: A projective Harnack curve of degree 8 with an hyperbolic node.

Definition 3.1. Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve with a single hyperbolic node. The node of \mathcal{C} is said to be next to ν if ν is the smooth vertex of its Newton polygon given in the previous theorem.

Denote by $\mathcal{H}_{\Delta,\nu}$ the space of simple Harnack curve with Newton polygon Δ and with a single hyperbolic node next to ν .

The space of simple Harnack curves $\mathcal{H}_{\Delta,\nu}$ can be seen as a subset of the projective space of polynomials supported inside Δ . Hence, $\mathcal{H}_{\Delta,\nu} \subset \mathbb{RP}^{k-1}$ where k is the number of integer points of Δ .

Theorem 4. The spine of a curve in $\mathcal{H}_{\Delta,\nu}$ is a tropical curve in $\mathbb{T}\mathcal{H}_{\Delta,\nu}$. The induced map

$$\mathcal{S} : \mathcal{H}_{\Delta,\nu} \rightarrow \mathbb{T}\mathcal{H}_{\Delta,\nu}$$

is a local diffeomorphism.

Remark. The only possible singularities of a simple Harnack curve in $\overline{\mathcal{H}_{\Delta,\nu}}$ are either real isolated double points or a unique cuspidal point. Indeed, the local coordinates given by the above theorem allows to perform contraction of ovals by contraction of holes of the spine. It gives rise to real isolated double points in the limit or a cusp, depending whether one contracts some of the smooth holes or the one next to the tropical node.

Remark. The map \mathcal{S} is not a global diffeomorphism, even on each connected component of the source, as it is not surjective. Indeed, performing contractions of holes at the tropical level, the corresponding ovals of a simple Harnack curve upstairs contract before the tropical ones. It implies in particular that the map \mathcal{S} does not extends continuously at the boundary of $\overline{\mathcal{H}_{\Delta,\nu}}$. Nevertheless, it is an interesting question to determine the image of \mathcal{S} , even in the smooth case, and whether it is a diffeomorphism on it. Note that the variational principle used in [KO06] to parametrized simple Harnack curves by the area of the holes of their amoeba still holds. The only difficulty to carry this approach to the present case lies in the study of the rational case.

3.2 Some technical conventions

In the rest of the paper, $\mathcal{C} \subset \mathcal{T}_\Delta$ will be a simple Harnack curve with a single hyperbolic node p . We denote by φ the connected component of \mathbb{RC}°

containing p , and by $\tilde{\varphi}$ its normalization in $\mathbb{R}\tilde{\mathcal{C}}^\circ$.

Up to toric transformation of $(\mathbb{C}^*)^2$, one can and do assume that Δ has an horizontal side supported on the x -axis. We will denote

$$b := |\partial\Delta \cap \mathbb{Z}^2| \quad \text{and} \quad g := |\text{Int}(\Delta) \cap \mathbb{Z}^2|$$

and refer to m as the number of sides of Δ .

We will always give $\partial\Delta$ the counter-clockwise orientation. It induces a cyclical order on the set of sides of Δ . We formalize it by a bijection

$$\begin{aligned} \mathbb{Z}/m\mathbb{Z} &\rightarrow \text{the set of sides of } \Delta \\ j &\mapsto s_j \end{aligned}$$

such that s_1 is the horizontal side of Δ supported on the x -axis. For each side s_j , $j \in \mathbb{Z}/m\mathbb{Z}$, there is a unique primitive integer vector v_j , $j \in \mathbb{Z}/m\mathbb{Z}$, supporting s_j and coherent with the orientation of $\partial\Delta$.

By lemma 4.2, we can and we do orient each connected component ϑ of $\mathbb{R}\tilde{\mathcal{C}}$ such that $\mathcal{A}(\mathbb{R}\tilde{\mathcal{C}}^\circ)$ is a locally concave parametrized curve.

For any u-oval ϑ , this orientation induces a cyclical order on the set (with possible repetitions) of the toric divisors \mathcal{D}_s as they are encountered by ϑ . We formalize it by a map

$$\begin{aligned} \mathbb{Z}/m_\vartheta\mathbb{Z} &\rightarrow \mathbb{Z}/m\mathbb{Z} \rightarrow \text{the set of sides of } \Delta \\ j &\mapsto \vartheta(j) \mapsto s_{\vartheta(j)} \end{aligned}$$

where m_ϑ is the number of points of $\vartheta_\infty := \tilde{\mathcal{C}}_\infty \cap \vartheta$. Denote also $\vartheta^\circ := \vartheta \setminus \vartheta_\infty$. Note that the map on the left is not necessarily injective, and is defined up to translation. For our purpose, we do not need to specify this map any further. For two vectors u and v in the plane, we denote by $\angle(u, v)$ the measure of the oriented angle from u to v with values in $[0; 2\pi[$.

Definition 3.2. *The index of a u-oval ϑ of $\mathbb{R}\tilde{\mathcal{C}}$ is defined by*

$$\text{ind}(\vartheta) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}/m_\vartheta\mathbb{Z}} \angle(v_{\vartheta(j)}, v_{\vartheta(j+1)})$$

3.3 Proof of the main theorems

The proofs of theorems 3 and 4 go by the following results.

3.3.1 Topological maximality

Contrary to Mikhalkin's original definition, no topological constraint explicitly appears in definition 2.1, except that simple Harnack curves are always of type 1. In particular, it is a priori not guaranteed that simple Harnack curves are M -curves. In the present case, one has the following.

Definition 3.3. *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve. A connected component of $\mathbb{R}\tilde{\mathcal{C}}$ is called a u -oval if it intersects $\tilde{\mathcal{C}}_\infty$, and a b -oval otherwise.*

Proposition 3.4. *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve with a single hyperbolic node. Then, the normalization $\mathbb{R}\tilde{\mathcal{C}} \subset \tilde{\mathcal{C}}$ is an M -curve. Moreover, $\mathbb{R}\tilde{\mathcal{C}}$ is the union of $(g - 1)$ b -ovals and a single u -oval, where $(g - 1)$ is the genus of $\tilde{\mathcal{C}}$.*

The proof goes by an estimation of the contribution of each connected component of the real part to the logarithmic Gauss map, case by case. We postpone it to the appendix.

3.3.2 Lifted coamoebas

The latter proposition implies that the normalization of such simple Harnack curve admits the following decomposition

$$\tilde{\mathcal{C}} := \tilde{\mathcal{C}}^- \cup \mathbb{R}\tilde{\mathcal{C}} \cup \tilde{\mathcal{C}}^+$$

where $\tilde{\mathcal{C}}^-$ and $\tilde{\mathcal{C}}^+$ are exchanged by the complex conjugation σ on $\tilde{\mathcal{C}}$. As $\text{Arg} \circ \sigma = -\text{id} \circ \text{Arg}$, the map Arg is orientation preserving on one “half” of $\tilde{\mathcal{C}}$ and orientation reversing on the other. By convention, fix $\tilde{\mathcal{C}}^+$ to be the one where Arg is orientation preserving.

Let us also introduce the following notation : if ν is a smooth vertex of a Newton polygon Δ , denote by $\Delta_\nu \subset \Delta$ the polygonal domain obtained by removing the parallelogram of area 1 in the corresponding corner of Δ .

Proposition 3.5. *Let $\mathcal{C} \subset \mathcal{T}_\Delta$ be a simple Harnack curve with a single hyperbolic node next to ν . Then, the restriction to $\tilde{\mathcal{C}}^+$ of the argument map Arg lifts to the universal covering \mathbb{R}^2 of T . Moreover, its lift Arg_0 is a diffeomorphism and*

$$\overline{\text{Arg}_0(\tilde{\mathcal{C}}^+)} = \tau(\Delta_\nu)$$

where τ is the composition of a rotation by $-\pi/2$ and a homothety by π .

The proof goes as follows : first one sees that the boundary is a piecewise linear curve obtained by concatenation of the “side” of Δ and, regarding this constraint, one shows then that the only possibility for this curve to enclose a domain of area predicted by corollary 2.5 is the one described above. We postpone it to the appendix.

Remark. The idea of lifting coamoebas already appeared in [NP10]. Here, the latter proposition replaces the 2-to-1 property of the amoeba map used for the topological classification of smooth simple Harnack curves in [Mik00]. Despite it could have been used there, it cannot be applied in general (consider for instance the quintic of figure 5).

3.3.3 Spines

Up to a toric transformation, one can assume that Δ contains the three points $(0,0)$, $(0,1)$ and $(1,0)$ and that $\nu = (0,0)$. It induces the local compactification of $(\mathbb{C}^*)^2$ by \mathbb{C}^2 in \mathcal{T}_Δ , where the two toric divisors adjacent to ν are the two coordinate axis. By corollary 4.10, the two asymptotes of $\mathcal{A}(\varphi)$ are horizontal leftward and vertical downward. As before, let $p = (p_1, p_2) \in (\mathbb{C}^*)^2$ be the node of \mathcal{C} , and choose $\varepsilon_1, \varepsilon_2 > 0$ such that the point $(\log |p_1| + \varepsilon_1, \log |p_2| + \varepsilon_2)$ belongs to the compact connected component of the complement of $\mathcal{A}(\mathbb{R}\mathcal{C})$ delimited $\mathcal{A}(\varphi)$. Define the following sets

$$\begin{aligned} R &:= \{(x, y) \in \mathbb{R}^2 \mid x \leq \log |p_1| + \varepsilon_1, y \leq \log |p_2| + \varepsilon_2\}, \\ H &:= \{(x, y) \in \mathbb{R}^2 \mid x \leq \log |p_1| + \varepsilon_1, y = \log |p_2| + \varepsilon_2\}, \\ V &:= \{(x, y) \in \mathbb{R}^2 \mid x = \log |p_1| + \varepsilon_1, y \leq \log |p_2| + \varepsilon_2\}. \end{aligned}$$

Lemma 3.6. $\partial(\mathcal{A}(\mathcal{C}) \cap R^c) = \mathcal{A}(\mathbb{R}\mathcal{C}) \cap R^c$. $\mathcal{A}|_{\mathcal{C} \setminus \mathcal{A}^{-1}(R)}$ is at most 2-to-1. For any connected component C of $\mathcal{A}^{-1}(R)$ in the normalization $\tilde{\mathcal{C}}$, $\mathcal{A}|_C$ is at most 2-to-1.

Proof From the way R was chosen, \mathcal{A} is an embedding on each connected component of $\mathbb{R}\mathcal{C} \cap \mathcal{A}^{-1}(R^c)$. For any such component, its image by \mathcal{A} split R^c into two halves. The proof is then the same as the one of lemma 8 in [Mik00]. \square

Lemma 3.7. One has the following decomposition $\mathcal{C} \cap \mathcal{A}^{-1}(R) = \mathcal{C}_H \cup \mathcal{C}_V$ where \mathcal{C}_H and \mathcal{C}_V are two connected Riemann surfaces with boundary such that $\mathcal{A}(\partial\mathcal{C}_H) \subset H$ and $\mathcal{A}(\partial\mathcal{C}_V) \subset V$. Moreover, \mathcal{A} is at most 2-to-1 when restricted to \mathcal{C}_H or \mathcal{C}_V .

Proof Draw a curve in the interior of $\mathcal{A}(\mathcal{C}) \cap R^c$ that starts on $\mathcal{A}(\varphi)$ and ends on the boundary of a non compact component of the complement of $\mathcal{A}(\mathcal{C})^c$. By the above 2-to-1 property, its lift in $\tilde{\mathcal{C}}$ is a topological circle ρ invariant under the complex conjugation that cuts the unique u -oval of $\tilde{\mathcal{C}}$ in two connected components. Indeed, the single u -oval of $\mathbb{R}\tilde{\mathcal{C}}$ intersects $\mathcal{A}^{-1}(R)$ in exactly two connected components α_H and α_V intersecting respectively only $\mathcal{A}^{-1}(H)$ and only $\mathcal{A}^{-1}(V)$. It follows that ρ cuts $\tilde{\mathcal{C}}$ into two connected components intersecting either $\mathcal{A}^{-1}(H)$ or $\mathcal{A}^{-1}(V)$. Denote the intersection with $\mathcal{A}^{-1}(R)$ of these components \mathcal{C}_H and \mathcal{C}_V respectively. \mathcal{C}_H has a connected component containing α_H . The amoeba of any other other connected component of \mathcal{C}_H has to have a non compact complement component in R , bounded by an arc joining H to V , contradicting the fact the ρ splits $\tilde{\mathcal{C}}$. The same argument applies to \mathcal{C}_V and we conclude that both \mathcal{C}_H and \mathcal{C}_V are connected.

The proof of the 2-to-1 property goes the same way as before. \square

Lemma 3.8. *There exists two functions g and h holomorphic on $\mathcal{A}^{-1}(R)$ such that C_V (resp. C_H) is the zero set of g (resp. h) and $g \cdot h = f$ where f is a polynomial defining \mathcal{C} .*

Proof The closure of $\mathcal{A}^{-1}(R)$ in \mathcal{T}_Δ is a polydisc D centred at the origin, in the local compactification \mathbb{C}^2 of $(\mathbb{C}^*)^2$. Let us consider an irreducible component C of C_V or C_H . By the classical implicit function theorem, there exists an open covering $D \subset \bigcup_j \mathcal{U}_j$ and a collection of function g_j holomorphic on \mathcal{U}_j such that the zero set of g_j is exactly $\mathcal{U}_j \cap C$. Hence the quotient g_j/g_k is a nowhere vanishing holomorphic function on the overlap $\mathcal{U}_j \cap \mathcal{U}_k$. We are looking for a global function g holomorphic on D such that its zero set is exactly C or equivalently such that g/g_j is a nowhere vanishing holomorphic function on \mathcal{U}_j . This amounts to solve the second Cousin problem, in the holomorphic case. As D is a Stein manifold, such that $H^2(D, \mathbb{Z}) = 0$, the result follows from theorem 7.4.4 in [Hö90]. \square

The local factorization of f on $\mathcal{A}^{-1}(R)$ induces a splitting of its associated

Ronkin function on R , that is

$$\begin{aligned}
N_f(x, y) &:= \frac{1}{(2i\pi)^2} \int_{\mathcal{A}^{-1}(x, y)} \frac{\log |f(z, w)|}{zw} dz \wedge dw \\
&= \frac{1}{(2i\pi)^2} \int_{\mathcal{A}^{-1}(x, y)} \frac{\log |g(z, w)| + \log |h(z, w)|}{zw} dz \wedge dw \\
&=: N_g(x, y) + N_h(x, y).
\end{aligned}$$

To each of the latter Ronkin functions, one can associate their respective spines \mathcal{S}_g and \mathcal{S}_h , as in [PR04]. These are two tropical curves in the domain R such that the amoeba $\mathcal{A}(\mathcal{C}_V)$ (*resp.* $\mathcal{A}(\mathcal{C}_H)$) deformation retracts on \mathcal{S}_g (*resp.* \mathcal{S}_h), see theorem 1.14.

Recall that A_f is the set of connected components of the complement of $\mathcal{A}(\mathcal{C})$. Similarly, denote A_h (*resp.* A_g) the set of connected components of the complement of $\mathcal{A}(\mathcal{C}_H)$ (*resp.* $\mathcal{A}(\mathcal{C}_V)$) in R . The same properties hold for N_h and N_g and \mathcal{S}_g and \mathcal{S}_h are defined the exact same way as \mathcal{S}_f . If one denotes by A_f^R the elements of A_f intersecting R , convexity implies that

$$\begin{aligned}
(S_f)|_R &= \max_{\alpha \in A_f^R} N_f^\alpha \\
&= \max_{\alpha \in A_f^R} N_h^\alpha + \max_{\alpha \in A_f^R} N_g^\alpha \\
&\leq \max_{\alpha \in A_h} N_h^\alpha + \max_{\alpha \in A_g} N_g^\alpha \\
&= S_h + S_g,
\end{aligned}$$

where the elements of A_f^R are seen as subsets of the elements of A_h and A_g in the second equality. There is equality if and only if the maps $A_f^R \rightarrow A_h$ and $A_f^R \rightarrow A_g$ given by the inclusion are both surjective, or equivalently no connected component of the complement of $\mathcal{A}(\mathcal{C}_H)$ is included in $\mathcal{A}(\mathcal{C}_V)$ and vice versa. Note that in such case

$$\mathcal{S}_f \cap R = \mathcal{S}_h \cup \mathcal{S}_g.$$

Lemma 3.9. *The stable intersection of \mathcal{S}_g and \mathcal{S}_h in R is 1.*

Proof Deform the tautological embedding $\iota_1 : \mathcal{C}_H \hookrightarrow (\mathbb{C}^*)^2$ in a family $\{\iota_t\}_{t \geq 1}$ such that $\mathcal{A} \circ \iota_t$ is the homothety of ratio $1/t$ centered at \mathcal{S}_h along the leaves of a foliation of $\mathcal{A}(\mathcal{C}_H)$ as in figure 6 of [Mik04], and such that $\text{Arg} \circ \iota_t = \text{Arg} \circ \iota_1$. Deform similarly \mathcal{C}_V and denotes $\mathcal{C}_{H,t}$ and $\mathcal{C}_{V,t}$ the deformations at a time t . As $\mathcal{A}(\mathcal{C}_{V,t}) \cap H = \mathcal{A}(\mathcal{C}_{H,t}) \cap V = \emptyset$, one has $\mathcal{C}_{H,t} \cap \mathcal{C}_{V,t} = 1$ for all t . Let us show on the other side that these intersection numbers converge to the stable intersection of \mathcal{S}_g and \mathcal{S}_h . Up to a small translation, assume that they intersect transversally, away from their vertices. The 2-to-1 property of lemma 3.6 implies that, near an intersection point p of \mathcal{S}_g and \mathcal{S}_h , $\mathcal{C}_{H,t}$ and $\mathcal{C}_{V,t}$ are topological cylinders, for t large enough. Their local intersection multiplicity is given by the intersection of the corresponding classes in $H_1((\mathbb{C}^*)^2, \mathbb{Z})$. Each class is given by the difference of the indices of the complement components of their respective amoebas, see the proof of lemma 11 in [Mik00]. Hence, it is also the local intersection multiplicity of \mathcal{S}_g and \mathcal{S}_h at p . \square

Lemma 3.10. *\mathcal{S}_g and \mathcal{S}_h are trees. Their Newton polygons are either segments of integer length 1 or triangles without inner integer points.*

Proof \mathcal{S}_h has exactly one vertical leaf going downward, and \mathcal{S}_g has exactly one horizontal leaf going leftward. Indeed, they have at least one by assumption, and cannot have more otherwise the stable intersection of \mathcal{S}_h and \mathcal{S}_g would be greater than 1, contradicting the previous lemma. If Δ_h and Δ_g are the respective Newton polygons, the latter implies that Δ_h is bounded from below by an horizontal side s_H of length 1 and Δ_g is bounded from the left by a vertical side s_V of length 1. For Δ_g , the side attached at the top of s_V , if there is, is strictly slanted toward the right, otherwise the corresponding leaf of \mathcal{S}_g would intersect H . The side attached at the bottom of s_V , if there is, is horizontal or strictly slanted toward the left, otherwise the corresponding leaf of \mathcal{S}_g would intersect the vertical leaf of \mathcal{S}_h , up to translation, contradicting the previous lemma. The only possibility is that Δ_g is either a binomial or a right angled triangle with integer height 1. The same arguments apply for Δ_h . \square

Proof of theorem 4 By the previous lemma, every connected component of the complement of $\mathcal{A}(\mathcal{C}_H)$ and $\mathcal{A}(\mathcal{C}_V)$ are unbounded in R . It implies that no connected components of the complement of $\mathcal{A}(\mathcal{C}_H)$ is hidden by $\mathcal{A}(\mathcal{C}_V)$

and vice versa. By the previous remarks, it implies that

$$\mathcal{S}_f \cap R = \mathcal{S}_h \cup \mathcal{S}_g.$$

It implies also that $\mathcal{A}(\mathcal{C})$ has exactly g visible holes, which is the maximal possible. Indeed, the previous lemma implies that none of the $(g-1)$ b-ovals of $\mathbb{R}\tilde{\mathcal{C}}$ intersect $\mathcal{A}^{-1}(R)$, and \mathcal{A} is at most 2-to-1 on this space, by lemma 3.6. Hence, their image by \mathcal{A} bounds a compact component of the complement. The same holds for the singular loop of $\mathcal{A}(\varphi)$. Hence, the complement of \mathcal{S}_f has also g compact connected components. By lemma 3.9, \mathcal{S}_f possess an hyperbolic node. It follows that \mathcal{S}_f is an element of $\mathbb{TH}_{\Delta,\nu}$. The fact that the map \mathcal{S} is local diffeomorphism results from the arguments of proposition 6 in [KO06]. In the present case, the coordinates on the target space are the intercept of the $(g-1)$ compact ovals of the normalization, plus the boundary coordinates. \square

Proof of theorem 3 The theorem can be deduced from the existence of a deformation retraction of $\mathcal{A}(\mathcal{C})$ onto \mathcal{S}_f such that $\mathcal{A}(\mathcal{C}_H)$ retracts on \mathcal{S}_h and $\mathcal{A}(\mathcal{C}_V)$ retracts on \mathcal{S}_g . Indeed, using the 2-to-1 property of lemma 3.6, such deformation is equivalent to an isotopy of the map $\mathcal{A} : \tilde{\mathcal{C}}/\sigma \rightarrow \mathbb{R}^2$ to the immersion of a ribbon R around the normalization of $\mathcal{S}(\mathcal{C})$ (see section 2.2). Hence, the topological type of \mathcal{C} can be recovered from $\mathcal{S}(\mathcal{C})$ by the unfolding procedure described in section 2.2.

The deformations of $\mathcal{A}(\mathcal{C}_H)$ and $\mathcal{A}(\mathcal{C}_V)$ have been constructed in the proof of lemma 3.9 and deformations of $\mathcal{A}(\mathcal{C}) \cap R^c$ onto $\mathcal{S}_f \cap R^c$ exist by classical theory (see theorem 1 in [PR04]). From this, one can easily construct a global deformation of $\mathcal{A}(\mathcal{C})$ with the required properties. \square

4 Appendix

4.1 Logarithmic geometry of planar curves (continuation)

Lemma 4.1. *Let s be a side of Δ and $(a, b) \in \mathbb{Z}^2$ a primitive integer vector supporting s . For any curve $\mathcal{C} \subset \mathcal{T}_\Delta$, and any point $p \in \mathcal{D}_s \cap \mathcal{C}$, one has*

$$\gamma(p) = [a : b].$$

Proof Suppose first that neither a nor b is zero. By the implicit function theorem, for any local coordinate t of \mathcal{C} centred at p , there exists 2 holomorphic functions $h_z(t)$ and $h_w(t)$ having a simple zero at the origin and a positive integer m such that

$$z(t) = (h_z(t))^{-bm} \text{ and } w(t) = (h_w(t))^{am}.$$

The number m is exactly the intersection multiplicity $\mathcal{D}_e \cap \mathcal{C}$ at p . By 1.4,

$$\begin{aligned} \gamma(p) &= \lim_{t \rightarrow 0} [-d \log(w(t)) : d \log(z(t))] \\ &= \lim_{t \rightarrow 0} \left[-am \cdot \frac{h'_w(t)}{h_w(t)} : -bm \cdot \frac{h'_z(t)}{h_z(t)} \right] \\ &= [a : b]. \end{aligned}$$

If a (resp. b) is zero, h_w (resp. h_z) is a non vanishing holomorphic function. The same computation leads to the result. \square

Lemma 4.2. *One has the following*

- * $\tilde{\mathcal{C}}_\infty \subset \tilde{F}$.
- * $\tilde{F} \subset \tilde{\mathcal{C}}$ is smooth if and only if $\tilde{\gamma}$ has no branching point on \mathbb{RP}^1 . In this case, \tilde{F} is a disjoint union of smoothly embedded circle in $\tilde{\mathcal{C}}$.
- * If \tilde{F} is smooth, $\tilde{\mathcal{O}}$ is a connected component of \tilde{F} and $\mathcal{A}_{|\tilde{\mathcal{O}}}$ is non constant on $\tilde{\mathcal{O}}$, then the logarithmic Gauss map of the parametrized curve $\mathcal{A}_{|\tilde{\mathcal{O}}}$ is given by the restriction of $\tilde{\gamma}$. In particular, it is monotonic and $\mathcal{A}_{|\tilde{\mathcal{O}}}$ has no inflection point.
- * Under the same assumptions, the latter statement holds for $\text{Arg}_{|\tilde{\mathcal{O}}}$.

Proof The first point is a direct consequence of lemmas 4.1 and 1.5. For the second point, if $\tilde{\gamma}$ has no branching point on \mathbb{RP}^1 , $\tilde{\gamma}|_{\tilde{F}}$ is a local diffeomorphism, and \tilde{F} is a topological covering of \mathbb{RP}^1 . It implies that \tilde{F} is a disjoint union of smoothly embedded circle in $\tilde{\mathcal{C}}$. If $\tilde{\gamma}$ has a branching point $q \in \mathbb{RP}^1$, then there exists $p \in \tilde{\gamma}^{-1}(q)$ such that \tilde{F} near p is diffeomorphic to the preimage of $\mathbb{R} \subset \mathbb{C}$ by $z \mapsto z^n$ for some $n \geq 2$. Hence, \tilde{F} is not

smooth at p . The last two points fall from the geometric interpretation of the logarithmic Gauss map. First, notice that \mathcal{A} and Arg are analytic on $\tilde{\mathcal{O}}$. Hence, they are either constant or locally injective. In the latter case, they admit a logarithmic Gauss map. Now, consider the tangent bundle of \mathbb{C}^2 restricted to the coordinate wise complex logarithm $Log(\tilde{\mathcal{C}}^\circ)$. Considering real and imaginray parts gives a splitting $\mathbb{R}^2 \oplus i\mathbb{R}^2$ of the latter bundle. We have seen in the previous lemma that the tangent bundle of $Log(\tilde{\mathcal{C}}^\circ)$ splits in a direct sum of two line bundles while restricted to $Log(\tilde{F}^\circ)$. One of them is contained in the \mathbb{R}^2 factor of the previous splitting, and the other one is contained in the $i\mathbb{R}^2$ factor. Denote them by \mathcal{L}_{Re} and \mathcal{L}_{Im} respectively. Note that the maps \mathcal{A} and Arg are just linear projections on \mathbb{R}^2 and $i\mathbb{R}^2$ in these logarithmic coordinates. Therefore, \mathcal{A} (*resp.* Arg) maps \mathcal{L}_{Re} (*resp.* \mathcal{L}_{Im}) to the tangent line bundle of $\mathcal{A}(\tilde{\mathcal{O}})$ (*resp.* $Arg(\tilde{\mathcal{O}})$). It follows that the Gauss maps of $\mathcal{A}|_{\tilde{\mathcal{O}}}$ and $Arg|_{\tilde{\mathcal{O}}}$ are both given by $\tilde{\gamma}$. By assumption, $\tilde{\gamma}$ has no critical point, that is $\mathcal{A}|_{\tilde{\mathcal{O}}}$ and $Arg|_{\tilde{\mathcal{O}}}$ have no inflection point. \square

We end up this section with a short description of the maps \mathcal{A} and Arg near the points of \mathcal{C}_∞ .

By convexity and finiteness of the area of $\mathcal{A}(\mathcal{C})$, one deduces that a neighbourhood of any point of \mathcal{C}_∞ is mapped by \mathcal{A} onto a thin tentacle going off to infinity along a certain asymptotic direction. If this point belongs to \mathcal{D}_s , lemma 4.1 implicitly states that this direction is orthogonal to the corresponding side s of Δ .

In the case of Arg , define $\tilde{\mathcal{C}}_{Arg}$ to be the real oriented blow-up of $\tilde{\mathcal{C}}$ at every point of $\tilde{\mathcal{C}}_\infty$. Denote by $S_p \subset \tilde{\mathcal{C}}_{Arg}$ the fiber of the blow-up over $p \in \tilde{\mathcal{C}}_\infty$.

Lemma 4.3. *The map Arg extends to $\tilde{\mathcal{C}}_{Arg}$. Moreover, if s is a side of Δ , (a, b) a primitive integer vector supporting s , and p belongs to \mathcal{D}_s , then $Arg : S_p \rightarrow T$ is an m -covering over a geodesic of slope $(-b, a)$, where m is the intersection multiplicity of $\tilde{\mathcal{C}}_\infty \cap \mathcal{D}_s$ at p .*

Proof As in the proof of lemma 4.1, use a local coordinate t and consider the expressions

$$z(t) = (h_z(t))^{-bm} \text{ and } w(t) = (h_w(t))^{am}.$$

For $t = re^{i\theta}$, one has

$$z(t) = r^{-bm} (z_0 e^{-ibm\theta} + o(1)) \text{ and } w(t) = r^{am} (w_0 e^{iam\theta} + o(1)).$$

It follows that for any $e^{i\theta} \in S^1$

$$\lim_{r \rightarrow 0} \arg(h_z(re^{i\theta})) = \arg(z_0) - bm \cdot \theta$$

and

$$\lim_{r \rightarrow 0} \arg(h_w(re^{i\theta})) = \arg(w_0) + am \cdot \theta.$$

This gives the result when a and b are non zero. Otherwise, replace $z(t)$ or $w(t)$ by a non vanishing function and repeat the same computation, see 4.1. \square

4.2 Proof of proposition 3.4

Lemma 4.4. *The only possible cases are the following*

- (α) φ is a self-intersecting arc in $(\mathbb{R}^*)^2$ joining the toric divisor \mathcal{D}_{s_j} to the toric divisor \mathcal{D}_{s_k} , with $\pi < \angle(v_j, v_k) < 2\pi$,
- (β) φ is a self-intersecting arc in $(\mathbb{R}^*)^2$ joining the toric divisor \mathcal{D}_{s_j} to the toric divisor \mathcal{D}_{s_k} , with $0 \leq \angle(v_j, v_k) < \pi$,
- (γ) φ is an immersed closed curve of rotational index 2 self-intersecting at p ,
- (δ) φ is the union of 2 arcs intersecting transversally at p .

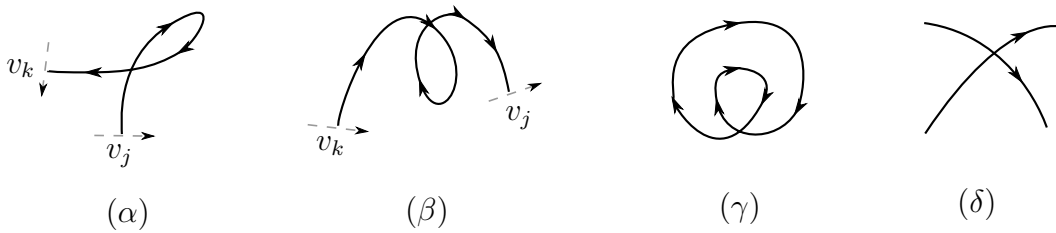


Figure 10: $\mathcal{A}(\varphi)$ in the 4 cases of lemma 4.4

Remark. The distinction we made between case (α) and (β) is not of topological nature *a priori*, but will be motivated later.

Proof Suppose the normalization $\tilde{\varphi}$ of φ is connected. Then it is either an open segment or a topological circle. The first possibility is split between (α) and (β) . For the second possibility, φ is an immersed circle self-intersecting at p . One of the 2 possible smoothings of φ gives 2 disjoint circles. If these circle are nested in the plane, then it corresponds to (γ) . If they are not, then φ is isotopic to the figure “ ∞ ” and has inflection. This contradicts lemma 4.2. Suppose now that the normalization $\tilde{\varphi}$ of φ is not connected. Then it is the union of 2 open segments, but this is case (δ) . \square

The classification of lemma 4.4 is relevant while computing the contribution to $\tilde{\gamma}$ of the ovals of $\mathbb{R}\tilde{\mathcal{C}}$. As the total contribution is constrained by the total reality of $\tilde{\gamma}$, it will prevent most of the cases to occur.

Remark. All the cases of the list in lemma 4.4 can appear for simple Harnack curve. Nevertheless, their manifestation forces the curve to have some other singularities, see figure 11. The case (α) was already illustrated by the cubic of figure 5.

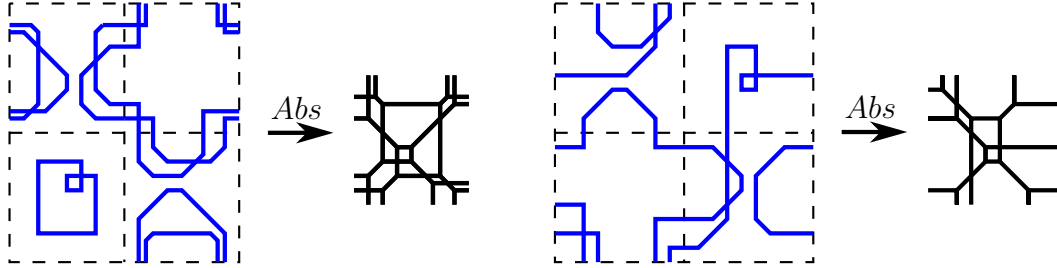


Figure 11: The case (β) on the left, and (γ) on the right. The case (δ) appears in both.

Lemma 4.5. *For a u-oval $\vartheta \subset \mathbb{R}\tilde{\mathcal{C}}$ such that \mathcal{A} is an embedding on each connected component of ϑ° , one has*

$$\deg \gamma|_{\vartheta} = |\vartheta_\infty| - 2 \cdot \text{ind}(\vartheta).$$

For a b-oval $\vartheta \subset \mathbb{R}\tilde{\mathcal{C}}$ for which \mathcal{A} is an embedding, one has

$$\deg \gamma|_{\vartheta} = 2.$$

Proof To prove the first formula, one has to compute the contribution on every connected component of ϑ° . As \mathcal{A} is an embedding, lemma 4.1 implies first that $0 \leq \angle(v_{\vartheta(j)}, v_{\vartheta(j+1)}) \leq \pi$ for each $j \in \mathbb{Z}/m_\vartheta\mathbb{Z}$ and that the contribution between the j -th and $(j+1)$ -th point of ϑ_∞ is given by

$$\frac{1}{\pi} \angle(-v_{\vartheta(j+1)}, v_{\vartheta(j)}) = \frac{1}{\pi} \left(\pi - \angle(v_{\vartheta(j)}, v_{\vartheta(j+1)}) \right) \geq 0,$$

according to our orientation convention. Summing over all j 's gives the desired formula. The second formula is the projective reformulation of the fact that a simple closed curve in the plane has rotational index 1. \square

Lemma 4.6. *In the cases (α) , and (β) of proposition 4.4, let ϑ be the unique u -oval in $\mathbb{R}\tilde{\mathcal{C}}$ containing $\tilde{\varphi}$. Then, one has*

$$\deg \gamma|_{\vartheta} = |\vartheta_\infty| + 2 - 2 \cdot \text{ind}(\vartheta).$$

In the case (γ) of proposition 4.4, let ϑ be the unique b -oval in $\mathbb{R}\tilde{\mathcal{C}}$ containing the node p . Then, one has

$$\deg \gamma|_{\vartheta} = 4.$$

Proof In the cases (α) , and (β) of proposition 4.4, the proof goes as the one of the first formula of the previous lemma, except that the contribution of the arc $\tilde{\varphi}$ is given for some j by

$$\frac{1}{\pi} \left(\pi - \angle(v_{\vartheta(j)}, v_{\vartheta(j+1)}) \right) + 2.$$

For the case (γ) , $\mathcal{A}(\vartheta)$ has rotational index 2 in the plane, that is 4 projectively. \square

Proof of proposition 3.4. By lemma 1.4, one has that

$$\deg \tilde{\gamma} = 2(g-1) - 2 + b = 2g + b - 4.$$

Consider first the cases (α) and (β) of proposition 4.4, and denote by ϑ the

u-oval of $\mathbb{R}\tilde{\mathcal{C}}$ containing $\tilde{\varphi}$. Let us compute

$$\begin{aligned}
\deg \tilde{\gamma} &= \sum_{\substack{\mathcal{O} \text{ oval} \\ \text{of } \mathbb{R}\tilde{\mathcal{C}}}} \deg \tilde{\gamma}|_{\mathcal{O}} = \sum_{\mathcal{O} \text{ u-oval}} \deg \tilde{\gamma}|_{\mathcal{O}} + \sum_{\mathcal{O} \text{ b-oval}} \deg \tilde{\gamma}|_{\mathcal{O}} \\
&= \left(\deg \tilde{\gamma}|_{\vartheta} + \sum_{\substack{\mathcal{O} \text{ u-oval} \\ \mathcal{O} \neq \vartheta}} \deg \tilde{\gamma}|_{\mathcal{O}} \right) + \sum_{\mathcal{O} \text{ b-oval}} \deg \tilde{\gamma}|_{\mathcal{O}} \\
&= \left(|\vartheta_{\infty}| + 2 - 2 \cdot \text{ind}(\vartheta) + \sum_{\substack{\mathcal{O} \text{ u-oval} \\ \mathcal{O} \neq \vartheta}} (|\mathcal{O}_{\infty}| - 2 \cdot \text{ind}(\mathcal{O})) \right) \\
&\quad + 2 \# \{ \mathcal{O} \text{ b-oval} \}
\end{aligned}$$

by lemmas 4.5 and 4.6

$$\begin{aligned}
&= \left(b + 2 - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} \text{ind}(\mathcal{O}) \right) + 2 \left(b_0(\mathbb{R}\tilde{\mathcal{C}}) - \# \{ \mathcal{O} \text{ u-oval} \} \right) \\
&= 2 b_0(\mathbb{R}\tilde{\mathcal{C}}) + b + 2 - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} (\text{ind}(\mathcal{O}) + 1).
\end{aligned}$$

It follows that

$$2 b_0(\mathbb{R}\tilde{\mathcal{C}}) - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} (\text{ind}(\mathcal{O}) + 1) = 2g - 6. \quad (2)$$

Moreover, $\mathbb{R}\tilde{\mathcal{C}}$ is of type 1 inside of $\tilde{\mathcal{C}}$ which is of genus $(g - 1)$, hence $b_0(\mathbb{R}\tilde{\mathcal{C}})$ is constrained by

$$b_0(\mathbb{R}\tilde{\mathcal{C}}) \leq g \text{ and } b_0(\mathbb{R}\tilde{\mathcal{C}}) \equiv g \pmod{2}.$$

If $b_0(\mathbb{R}\tilde{\mathcal{C}}) = g - 2l$, then

$$\begin{aligned}
2g - 6 &= 2 b_0(\mathbb{R}\tilde{\mathcal{C}}) - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} (\text{ind}(\mathcal{O}) + 1) \\
&\leq 2g - 4l - 4 \# \{ \mathcal{O} \text{ u-oval} \} \\
&\leq 2g - 4(l + 1).
\end{aligned}$$

It implies that $l = 0$, $b_0(\mathbb{R}\tilde{\mathcal{C}}) = g$, and

$$\sum_{\mathcal{O} \text{ u-oval}} (ind(\mathcal{O}) + 1) = 3.$$

Hence ϑ is the unique u-oval and $ind(\vartheta) = 2$. We proved the result for cases (α) and (β) .

Consider now the case (γ) of proposition 4.4, and denote once again by ϑ the u-oval of $\mathbb{R}\tilde{\mathcal{C}}$ containing $\tilde{\varphi}$. We repeat the same computation

$$\begin{aligned} deg \tilde{\gamma} &= \sum_{\mathcal{O} \text{ u-oval}} deg \tilde{\gamma}|_{\mathcal{O}} + \sum_{\mathcal{O} \text{ b-oval}} deg \tilde{\gamma}|_{\mathcal{O}} \\ &= \sum_{\mathcal{O} \text{ u-oval}} (|\mathcal{O}_{\infty}| - 2 \cdot ind(\mathcal{O})) + \left(\sum_{\substack{\mathcal{O} \text{ u-oval} \\ \mathcal{O} \neq \vartheta}} 2 + 4 \right) \end{aligned}$$

as ϑ contributes to 4 according to 4.6

$$\begin{aligned} &= \left(b - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} ind(\mathcal{O}) \right) + \left(2 \# \{ \mathcal{O} \text{ b-oval} \} + 2 \right) \\ &= 2b_0(\mathbb{R}\tilde{\mathcal{C}}) + b + 2 - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} (ind(\mathcal{O}) + 1). \end{aligned}$$

Once again, we end up with equation (2). The same arguments as above imply that $b_0(\mathbb{R}\tilde{\mathcal{C}}) = g$, and that there is a unique u-oval ϑ with $ind(\vartheta) = 2$. We proved the result for case (γ) .

Consider finally the case (δ) of proposition 4.4. Note that every oval of $\mathbb{R}\tilde{\mathcal{C}}$ satisfies the assumptions of 4.5. We compute as before

$$\begin{aligned} deg \tilde{\gamma} &= \sum_{\mathcal{O} \text{ u-oval}} deg \tilde{\gamma}|_{\mathcal{O}} + \sum_{\mathcal{O} \text{ b-oval}} deg \tilde{\gamma}|_{\mathcal{O}} \\ &= \left(b - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} ind(\mathcal{O}) \right) + \left(2 \# \{ \mathcal{O} \text{ u-oval} \} \right) \\ &= 2b_0(\mathbb{R}\tilde{\mathcal{C}}) + b - 2 \cdot \sum_{\mathcal{O} \text{ u-oval}} (ind(\mathcal{O}) + 1). \end{aligned}$$

It follows that

$$2b_0(\mathbb{R}\tilde{\mathcal{C}}) - 2 \cdot \sum_{\substack{\mathcal{O} \text{ u-oval} \\ \mathcal{O} \neq \vartheta}} (ind(\mathcal{O}) + 1) = 2g - 4,$$

implying in turn that $b_0(\mathbb{R}\tilde{\mathcal{C}}) = g$, that there is a unique u-oval ϑ and that $\text{ind}(\vartheta) = 1$. Equivalently, the cyclical order on the b boundary points of \mathcal{C} induced by ϑ and the one induced by the boundary $\partial\Delta$ of the moment polygon Δ are the same. In such case, the image by \mathcal{A} of any two connected components of ϑ° intersect at 0 or 2 points, but \mathcal{C} has exactly one singular point. This is a contradiction. \square

Note that, along the latter proof, we obtained the following

Lemma 4.7. *For a simple Harnack curve with only one hyperbolic node, the case (δ) of lemma 4.4 cannot occur.*

4.3 Proof of proposition 3.5

Lemma 4.8. *The restriction to $\tilde{\mathcal{C}}^+$ of the argument map Arg lifts to the universal covering \mathbb{R}^2 of T . Moreover, its lift Arg_0 is a local diffeomorphism.*

Proof This lemma is a corollary of proposition 3.4. Indeed, the latter implies that $\tilde{\mathcal{C}}^+$ is homeomorphic to an open disc with exactly $(g - 1)$ holes. Compactifying $\tilde{\mathcal{C}}^+$ by attaching back $\mathbb{R}\tilde{\mathcal{C}}$, one sees that the fundamental group of $\tilde{\mathcal{C}}^+$ is generated by the $(g - 1)$ b-ovals of $\mathbb{R}\tilde{\mathcal{C}}$. The argument map contracts each of these ovals to one of the four points $\{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\}$ in T . In other words, the map

$$\text{Arg} : \pi_1(\tilde{\mathcal{C}}^+) \rightarrow \pi_1(T)$$

is trivial. This is the necessary and sufficient condition for Arg to lift to a map

$$\text{Arg}_0 : \tilde{\mathcal{C}}^+ \rightarrow \mathbb{R}^2.$$

By the definition of simple Harnack curve and lemma 1.5, Arg is a local diffeomorphism. So is Arg_0 . \square

Now, define the topological disc D as follows : consider first the closure of $\tilde{\mathcal{C}}^+$ in $\tilde{\mathcal{C}}_{\text{Arg}}$, see lemma 4.3. It is a closed disc with $(g - 1)$ open holes bounded by the b-ovals of $\mathbb{R}\tilde{\mathcal{C}}$. Now contract to a point every connected component of $\mathbb{R}\tilde{\mathcal{C}}_{\text{Arg}}$ that is in the closure of $\tilde{\mathcal{C}}^+$. The result is clearly a topological disc that we denote D .

Lemma 4.9. *Arg_0 extends to a differentiable map $\text{Arg}_0 : D \rightarrow \mathbb{R}^2$. Denote by ϑ the unique u-oval of $\tilde{\mathcal{C}}$. Then, Arg_0 maps the boundary of D to the*

piecewise linear curve with vertices in $\pi\mathbb{Z}^2$ obtained by the concatenation of the vectors $\tau(v_{\vartheta(j)})$ according to the cyclical ordering $j \in \mathbb{Z}/m_{\vartheta}\mathbb{Z}$, τ being defined in proposition 3.5.

Remark. The points of $Arg_0(D) \cap \pi\mathbb{Z}^2$ are exactly the points of D coming from the contracted connected components of $\mathbb{R}\tilde{\mathcal{C}}_{Arg}$.

Proof The fact that Arg_0 extends has been proven in lemma 4.3. Differentiability at the boundary is implicitly given in the proof of lemma 4.3. To see differentiability at the points obtained by contraction of the b-ovals, use the same arguments as in the proof of theorem 1.

The second part of the statement falls from lemma 4.3 and the way we defined $\tilde{\mathcal{C}}^+$. \square

Now, we are ready to prove the proposition 3.5. We have to solve a combinatorial isoperimetrical inequality problem. On one side, the area enclosed by the piecewise linear curve $Arg_0(\partial D)$ is given by theorem 1. On the other, the geometry and in particular the length of the piecewise linear curve $Arg_0(\partial D)$ is constraint by the previous lemma.

Proof of proposition 3.5 By theorem 1, one has that

$$\begin{aligned} Area(Arg_0(D)) &= Area(Arg_0(\tilde{\mathcal{C}}^+)) = \frac{1}{2}Area(Arg(\mathcal{C}^\circ)) \\ &= -\frac{\pi^2}{2}\chi(\tilde{\mathcal{C}}^\circ). \end{aligned}$$

In the present case, Khovanskii's formula [Kho78] gives

$$-\chi(\tilde{\mathcal{C}}^\circ) = (2g + b - 2) - 2.$$

Pick's formula gives in turn

$$Area(Arg_0(D)) = \pi^2((g + b/2 - 1) - 1) = \pi^2(Area(\Delta) - 1) \quad (3)$$

Consider the case (α) of lemma 4.4. By the previous lemma, $Arg_0(\partial D)$ is a piecewise linear curve with vertices in $\pi\mathbb{Z}^2$. As we have seen in the proof of lemma 4.5, it is locally convex everywhere except at the vertex coming from $\tilde{\varphi} \subset \tilde{\mathcal{C}}_{Arg}$, where by assumption the angle interior to $Arg_0(D)$ is strictly between π and 2π . Let $j \in \mathbb{Z}/m_{\vartheta}\mathbb{Z}$ be such that this non convex angle is

formed by $\tau(v_{\vartheta(j)})$ and $\tau(v_{\vartheta(j+1)})$. If one permutes these 2 vectors, the area of the domain of \mathbb{R}^2 enclosed by the piecewise linear curve increases at least by π^2 , and exactly by π^2 if and only if $v_{\vartheta(j)}$ and $v_{\vartheta(j+1)}$ span a parallelogram of area 1. Suppose the new polygonal domain that we just obtained is still not convex. Then, one can repeat the previous construction, and increase the area until we end up with a convex domain. This convex domain can be nothing but Δ (up to translation). This contradicts (3). Hence, the result of the permutation were already convex and $v_{\vartheta(j)}$ and $v_{\vartheta(j+1)}$ have to span a parallelogram of area 1, by (3). In other words, there exists a smooth vertex ν of Δ such that $Arg_0(D) = \tau(\Delta_\nu)$.

In cases (β) and (γ) of lemma 4.4, one has that $Arg_0(\partial\mathcal{D})$ is convex. In such case, its rotational index is exactly computed by $ind(\vartheta)$, where ϑ is the unique u-oval of $\tilde{\mathcal{C}}$. It has been shown in the proof of proposition 3.4 that this rotational index is 2. Then, there exists two distinct points p_1 and p_2 on ∂D mapped to the same point in \mathbb{R}^2 . They cut ∂D into two arcs γ_1 and γ_2 . Denote by Δ_1 and Δ_2 the polygonal domains enclosed by γ_1 and γ_2 respectively. Then

$$Area(Arg_0(D)) = Area(\Delta_1) + Area(\Delta_2). \quad (4)$$

Two cases can occurs : either p_1 and p_2 are mapped to a point of $\pi\mathbb{Z}^2$, or not.

In the first case, $\tau^{-1}(\Delta_1)$ and $\tau^{-1}(\Delta_2)$ are two polygonal domains in \mathbb{R}^2 with integer vertices. Reorder their edges in convex position in order to get two convex polygon \square_1 and \square_2 . Then, one has the Minkowski sum

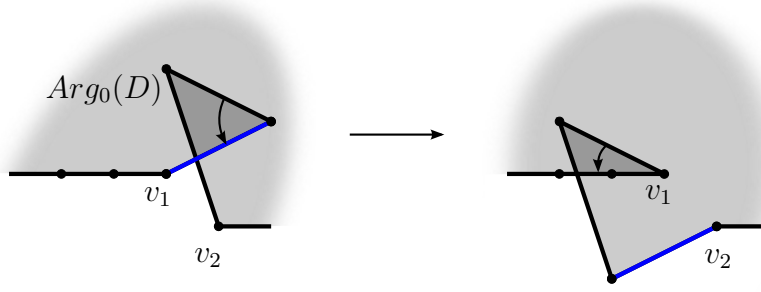
$$\Delta = \square_1 + \square_2. \quad (5)$$

By (3), (4), one has

$$Area(\Delta) - 1 \leq Area(\square_1) + Area(\square_2),$$

and (5) provide the opposite inequality. It means that the mixed volume $Vol(\square_1, \square_2) = 1$. By [Kus76], This is the intersection number of two generic curves of respective Newton polygon \square_1 and \square_2 . By (5), the union of two such curves has Newton polygon Δ . Such reducible curves form a component of the space of nodal curves in \mathcal{T}_Δ . By Horn parametrization, this space is irreducible. Hence, every nodal curve in \mathcal{T}_Δ is reducible, in particular, \mathcal{C} is. This is a contradiction.

In the second case, p_1 and p_2 are not mapped on a point of $\pi\mathbb{Z}^2$. Denote by v_1 the closest vertex of $Arg_0(\partial D)$ before the self-intersection point $Arg_0(p_1)$, and v_2 the closest vertex of $Arg_0(\partial D)$ after it. Now cut the first edge of $Arg_0(\partial D)$ after v_1 and past it before v_2 as shown below.



This construction has the following properties :

- (*) the area enclosed by the resulting curve is strictly greater than $Area(Arg_0(D))$,
- (*) the angle at the vertex next the self-intersection point strictly decreases, if it is still convex.

The latter property implies that, repeating this process, one has to end up either in the case where the self-intersection point is moved to a point of $\pi\mathbb{Z}^2$, then the previous treatment leads to a contradiction; or in the case of a piecewise linear curve of rotational index 1 that is convex except at one vertex. This amounts to the treatment of case (α) . The first property implies that

$$Area(Arg_0(D)) < \pi^2(Area(\Delta) - 1).$$

This is in contradiction with (3). By lemma 4.7, the case (δ) cannot occur. The theorem is proved. \square

Note that during the proof, we obtained the following

Corollary 4.10. *Only the case (α) of lemma 4.4 can occur.*

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